Final Exam Review Session.

1. Chain Rule.

$$f(x) = \ln\left(x + \sqrt{\cos x}\right)$$

$$\frac{df}{dx} = \frac{d\left(\ln\left(x+\sqrt{\cos x}\right)\right)}{d\left(x+\sqrt{\cos x}\right)} \cdot \frac{d\left(x+\sqrt{\cos x}\right)}{dx}$$

$$= \frac{1}{x+\sqrt{\cos x}} \cdot \left(\frac{dx}{dx} + \frac{d\left(\sqrt{\cos x}\right)}{dx}\right)$$

$$= \frac{1}{x+\sqrt{\cos x}} \cdot \left(1 + \frac{d\left(\sqrt{\cos x}\right)}{d(\cos x)} \cdot \frac{d(\cos x)}{dx}\right)$$

$$= \frac{1}{x+\sqrt{\cos x}} \cdot \left(1 + \frac{1}{2\sqrt{\cos x}} \cdot (-\sin x)\right)$$

$$= \frac{1}{x+\sqrt{\cos x}} \cdot \left(1 - \frac{\sin x}{2\sqrt{\cos x}}\right)$$

Test 3 Q3.

$$y = \ln\left(\frac{(3x-7)^6}{\sqrt{x^2+1}}\right)$$

$$= \ln\left((3x-7)^6\right) - \ln\left(\sqrt{x^2+1}\right)$$

$$= \ln\left((3x-7)^6\right) - \ln\left(\left(x^2+1\right)^{1/2}\right)$$

$$= 6\ln(3x-7) - \frac{1}{2}\ln\left(x^2+1\right)$$

$$y' = \left[6\,\boxed{\frac{d(\ln(3x-7))}{d(3x-7)}} \cdot \frac{d(3x-7)}{dx}\right]$$

$$- \left[\frac{1}{2}\,\frac{d\left(\ln\left(x^2+1\right)\right)}{d\left(x^2+1\right)} \cdot \frac{d\left(x^2+1\right)}{dx}\right]$$

$$= \left[6 \cdot \boxed{\frac{1}{3x-7}} \cdot 3\right] - \left[\frac{1}{\cancel{2}} \cdot \frac{1}{x^2+1} \cdot \cancel{2}x\right]$$

$$= \frac{18}{3x-7} - \frac{x}{x^2+1}$$

3. Find the equation of the plane containing the point $(3,5,-1)$ and contains the line

$$x = 4-t$$
$$y = 2t-1$$
$$z = -3t$$

$$\vec{r} = \langle 4,-1,0\rangle + t\langle -1,2,-3\rangle, \quad t \in \mathbb{R}.$$

$$\vec{v} = \langle -1,2,-3\rangle$$

Want to find another vector in the plane.

Since $\vec{r}$ lies in the plane then $(4,-1,0)$ is also a point on the plane.

$$\vec{w} = \langle 4-3, -1-5, 0+1\rangle = \langle 1,-6,1\rangle$$

$$\vec{w}\times\vec{v} = \left|\begin{array}{cccccc} 1 & -6 & 1 & 1 & -6 & 1 \\ -1 & 2 & -3 & -1 & 2 & -3 \end{array}\right|$$

$$\langle 18-2, -1+3, 2-6\rangle$$

$$= \langle 16,2,-4\rangle := n$$

$$\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0. \;]$$

$$\langle 16,2,-4\rangle \cdot \langle x-3, y-5, z+1\rangle = 0$$

$$16(x-3) + 2(y-5) - 4(z+1) = 0$$

$$16x + 2y - 4z - 48 - 10 - 4 = 0$$

$$16x + 2y - 4z = 62$$

$$8x + y - 2z = 31$$

$$\vec{r} = \vec{r}_0 + t\vec{v}, \quad t \in \mathbb{R}.$$

$$\vec{r} - \vec{r}_0 = t\vec{v}$$

$$\vec{n}\cdot t\vec{v} = \vec{n}\cdot(\vec{r}-\vec{r}_0) = 0$$

4. Find the equation of the plane passing through $A(2,1,2)$, $B(3,-8,6)$, $C(-2,-3,1)$.

$$\vec{AB} = \langle 3-2, -8-1, 6-2\rangle = \langle 1,-9,4\rangle$$

$$\vec{AC} = \langle -2-2, -3-1, 1-2\rangle = \langle -4,-4,-1\rangle$$

$$\vec{AB}\times\vec{AC} = \left|\begin{array}{cccccc} 1 & -9 & 4 & 1 & -9 & 4 \\ -4 & -4 & -1 & -4 & -4 & -1 \end{array}\right|$$

$$\langle 9+16, -16+1, -4-36\rangle$$

$$= \langle 25,-15,-40\rangle := \vec{n}$$

$$\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0$$

let r_0 be $\langle 2,1,2\rangle$

$$\langle 25,-15,-40\rangle \cdot \langle x-2, y-1, z-2\rangle = 0$$

$$25(x-2) - 15(y-1) - 40(z-2) = 0$$

$$25x - 15y - 40z - 50 + 15 + 80 = 0$$

$$25x - 15y - 40z + 45 = 0$$

$$5x - 3y - 8z + 9 = 0$$

5. Find the vector, parametric, and symmetric equations of the line passing through points $A(-8,1,4)$ and $B(3,-2,4)$.

We need:

- a start-point vector, $\vec{r_0}$
- a direction vector, $\vec{v}$

$$\vec{v} = \vec{AB} = \langle 3+8, -2-1, 4-4 \rangle$$
$$= \langle 11, -3, 0 \rangle$$

$$\vec{r_0} = \vec{OA} = \langle -8, 1, 4 \rangle.$$

$$\vec{r} = \vec{r_0} + t\vec{v}$$
$$= \langle -8, 1, 4 \rangle + t \langle 11, -3, 0 \rangle, \ t \in \mathbb{R}.$$

Parametric: $x = -8 + 11t$
$y = 1 - 3t$
$z = 4 + 0t = 4$

Symmetric: $\dfrac{x+8}{11} = \dfrac{y-1}{-3}, \quad z=4$

6. Find the eq'n of the line passing through $A(0, \frac{1}{2}, 1)$ and $B(2, 1, -3)$.

$$\vec{v} = \vec{AB} = \langle 2-0, 1-\tfrac{1}{2}, -3-1 \rangle$$
$$= \langle 2, \tfrac{1}{2}, -4 \rangle$$

$$\vec{r_0} = \vec{OA} = \langle 0, \tfrac{1}{2}, 1 \rangle$$

$$\vec{r} = \vec{r_0} + t\vec{v}$$
$$= \langle 0, \tfrac{1}{2}, 1 \rangle + t \langle 2, \tfrac{1}{2}, -4 \rangle, \ t \in \mathbb{R}.$$

Parametric: $x = 0 + 2t = 2t$
$y = \frac{1}{2} + \frac{1}{2}t$
$z = 1 - 4t$

Symmetric: $\dfrac{x}{2} = \dfrac{y-\frac{1}{2}}{1/2} = \dfrac{z-1}{-4}$

7.a. $(i \times j) \times k.$

$= k \times k = \vec{0}$

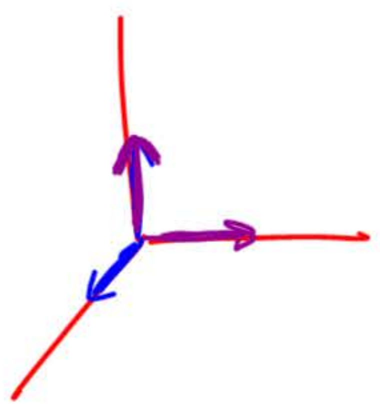
b. $k \times (i - 2j)$

$= (k \times i) + (k \times -2j)$

$= (k \times i) - 2(k \times j)$

$= j - 2(-i)$

$= j + 2i$



$i = \langle 1, 0, 0 \rangle$
$j = \langle 0, 1, 0 \rangle$
$k = \langle 0, 0, 1 \rangle$

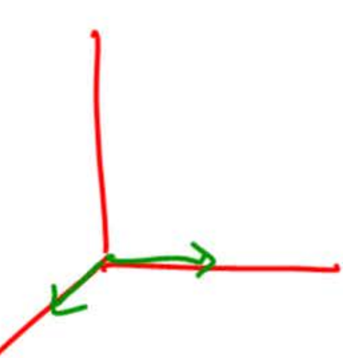
c. $(j-k) \times (k-i)$

$= (j \times (k-i)) - (k \times (k-i))$

$= (j \times k) - (j \times i) - (k \times k) + (k \times i)$

$= i + k - \vec{0} + j = i + j + k.$

d. $(i+j) \times (i-j)$

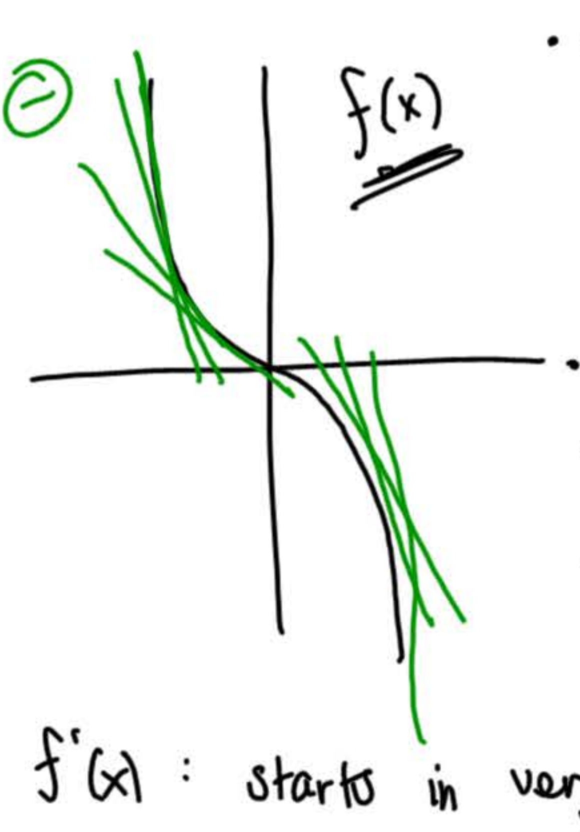


$= (i \times (i-j)) + (j \times (i-j))$

$= (i \times i) - (i \times j) + (j \times i) - (j \times j)$

$= \vec{0} - k - k - \vec{0}$

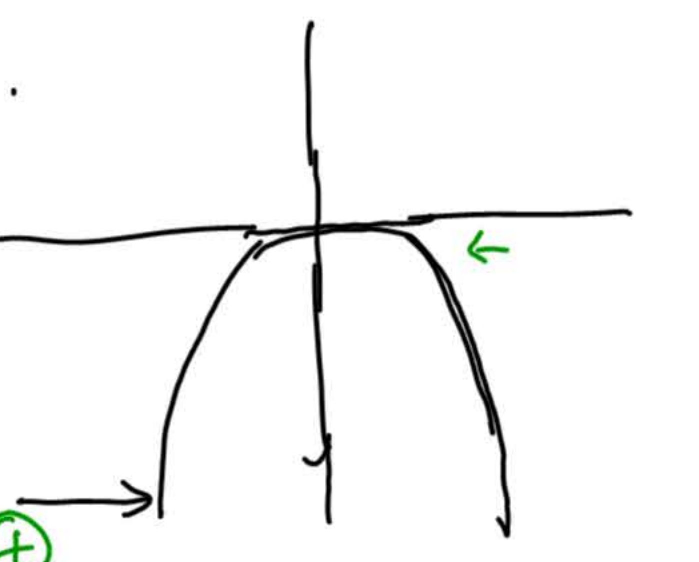
$= -2k.$



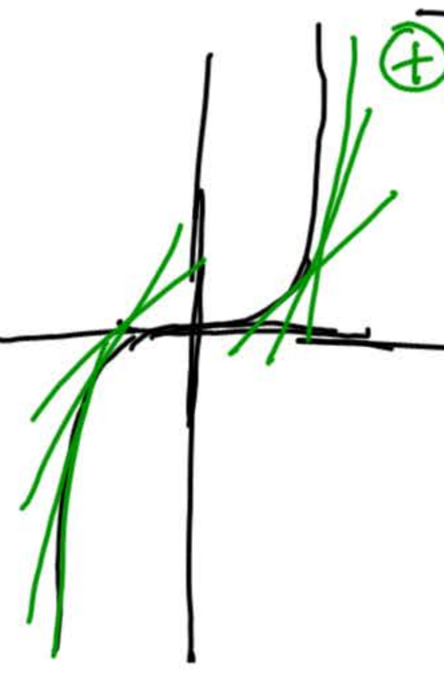
- on left, slope is negative. as x increases, slope is getting less negative.
- on right, slope is negative as increases, slope is getting more negative.

$f'(x)$: starts in very negative region, increases, goes back into negative region.

$f'(x)$ is graph C.



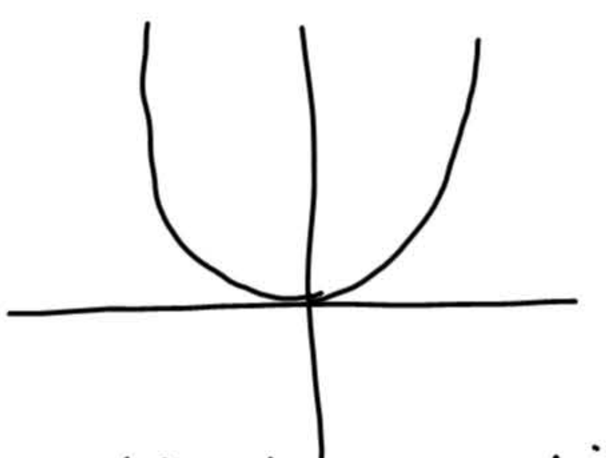
$f(x)$



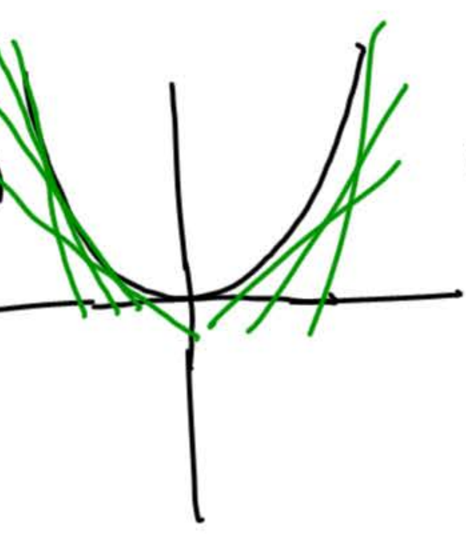
on left, all slopes positive, and slope is decreasing.

on right. all slopes positive and slope is increasing

$f'(x)$



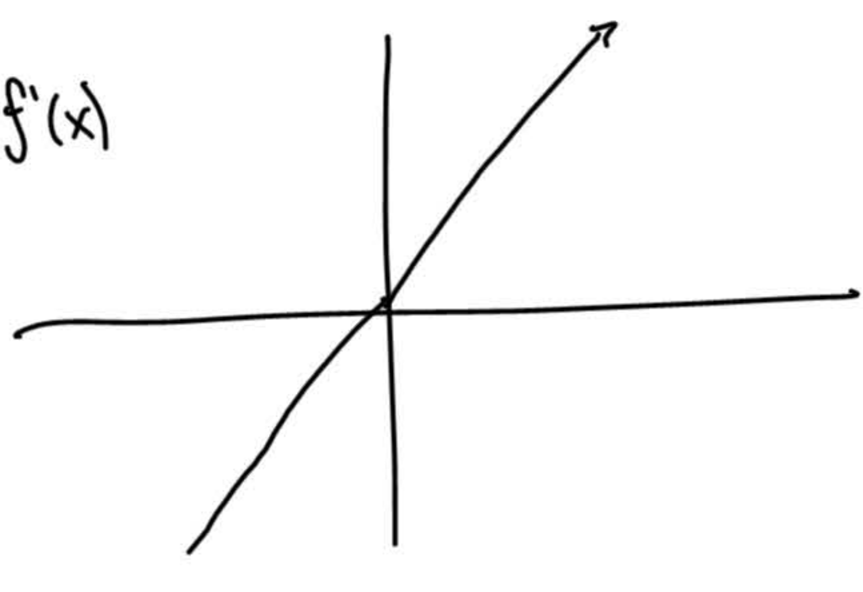
$f(x)$



on left, slope is negative, and getting less negative

on right, slope is positive and getting more positive

$f'(x)$



Parabola : $y = ax^2 + bx + c$

$y' = 2ax + b$] linear in x.

Find the equation of the tangent line of $y = \ln(x^3 - 7)$ at point $(2,0)$.

$$y' = \frac{d(\ln(x^3-7))}{d(x^3-7)} \cdot \frac{d(x^3-7)}{dx}$$

$$= \frac{1}{x^3-7} \cdot 3x^2$$

$$= \frac{3x^2}{x^3-7}$$

$$m = y'(2) = \frac{3(2)^2}{(2)^3-7} = \frac{12}{1} = 12$$

$x=2$, $y=0$, $m=12$, $b=?$

$y = mx + b$ $\qquad$ $y = 12x - 24$

$0 = (12)(2) + b$

$b = -24$

Determine the absolute maximum and minimum of $f(x) = \frac{\ln x - 1}{x}$ on $[1, 10]$. $x > 0$

$$f'(x) = \frac{(\frac{1}{x})x - (\ln x - 1)(1)}{x^2}$$

$$= \frac{1 - \ln x + 1}{x^2} = \frac{-\ln x + 2}{x^2}$$

Set $f'(x) = 0$.

$-\ln x + 2 = 0$

$\ln x = 2$

$x = e^2$

$$f(1) = \frac{\ln(1) - 1}{1} = -1$$

$$f(e^2) = \frac{\ln(e^2) - 1}{e^2} = \frac{1}{e^2} \approx 0.135$$

$$f(10) = \frac{\ln(10) - 1}{10} \approx 0.130$$

The absolute maximum is 0.135

The absolute minimum is -1.

Determine the intervals on which $f(x) = \frac{3+x^2}{x-1}$ is increasing.

$$f'(x) = \frac{(2x)(x-1) - (3+x^2)(1)}{(x-1)^2}$$

$$= \frac{2x^2 - 2x - 3 - x^2}{(x-1)^2}$$

$$= \frac{x^2 - 2x - 3}{(x-1)^2}$$

$$= \frac{(x-3)(x+1)}{(x-1)^2}$$

$f'(x) = 0$

$(x-3)(x+1) = 0$

$x = 3$ or $x = -1$

-1 $\qquad$ 3

	$x < -1$	$-1 < x < 3$	$x > 3$
$(x-3)(x+1)$	+	−	+
$(x-1)^2$	+	+	+
	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$

$f(x)$ is increasing when $f'(x) > 0$.

f increasing on intervals $(-\infty, -1) \cup (3, \infty)$.

Test 2 Q5a

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x^4+1}} = \lim_{x\to\infty} \frac{x^2}{\sqrt{x^4\left(1+\frac{1}{x^4}\right)}}$$

$$= \lim_{x\to\infty} \frac{x^2}{\sqrt{x^4}\sqrt{1+\frac{1}{x^4}}}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}}{\cancel{x^2}\sqrt{1+\frac{1}{x^4}}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^4}}}$$

$$= 1$$

$$f(x) = x(x-5)^{2/3}$$

$$f'(x) = \underbrace{(1)(x-5)^{2/3}}_{\uparrow} + \underbrace{(x)\left(\tfrac{2}{3}\right)(x-5)^{-\frac{1}{3}}}$$

$$= (x-5)^{2/3} \cdot \frac{3(x-5)^{1/3}}{3(x-5)^{1/3}} + \frac{2x}{3}\cdot(x-5)^{-\frac{1}{3}}$$

$$= \frac{3(x-5) + 2x}{3(x-5)^{1/3}} = \frac{3x-15+2x}{3(x-5)^{1/3}}$$

$$= \frac{5x-15}{3(x-5)^{1/3}} = \frac{5(x-3)}{3(x-5)^{1/3}}$$

Set $f'(x)=0$

$5(x-3)=0$

$x=3$

$f'(x)$ undefined when $x=5$. $x=5$ is in the domain of $f(x)$ so $x=5$ is also a critical number.

Final Exam Q 25.

$f'(0)=f'(2)=0$ ← horizontal tangents. $\underbrace{f(0)=-3}_{\text{y-intercept.}}$

$f'(x)<0$ when $x<0$ ← f is decreasing

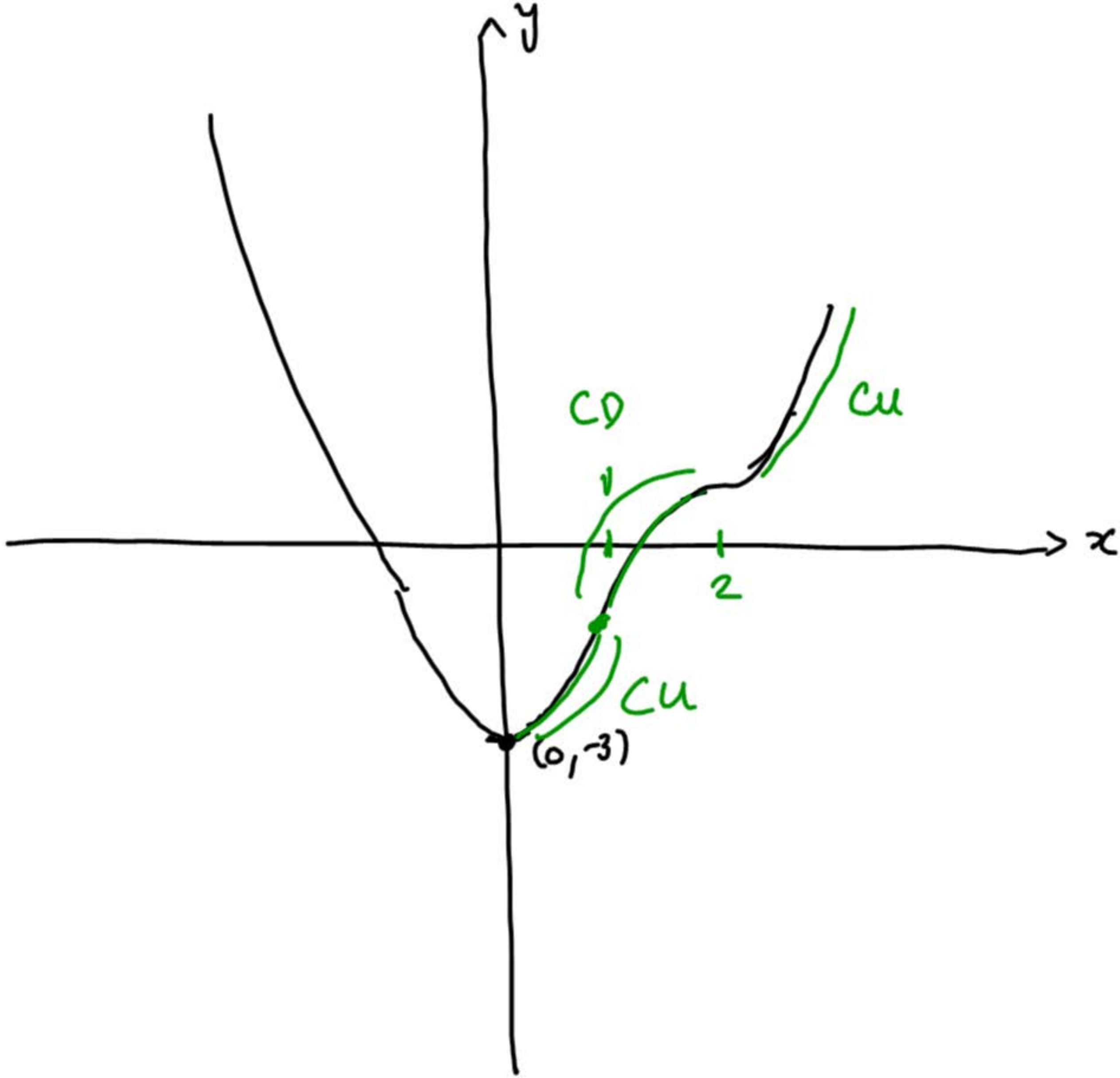
$f'(x)>0$ when $0<x<2$, $x>2$

← f is increasing

$f''(1)=f''(2)=0$ ← possible inflection points

$f''(x)>0$ when $x<1$, $x>2$ ← concave up.

$f''(x)<0$ when $1<x<2$ ← concave down.


y
x
CD
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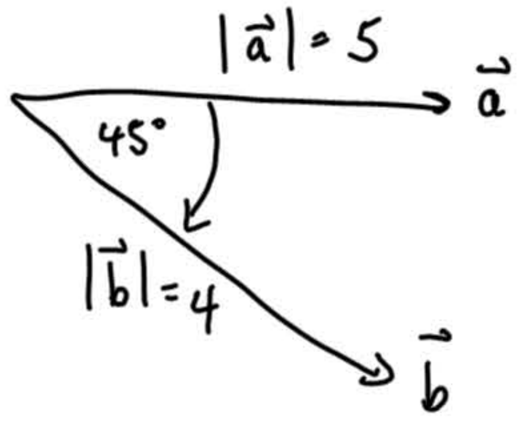
CU
1
2
(0, -3)

A12 Q3.

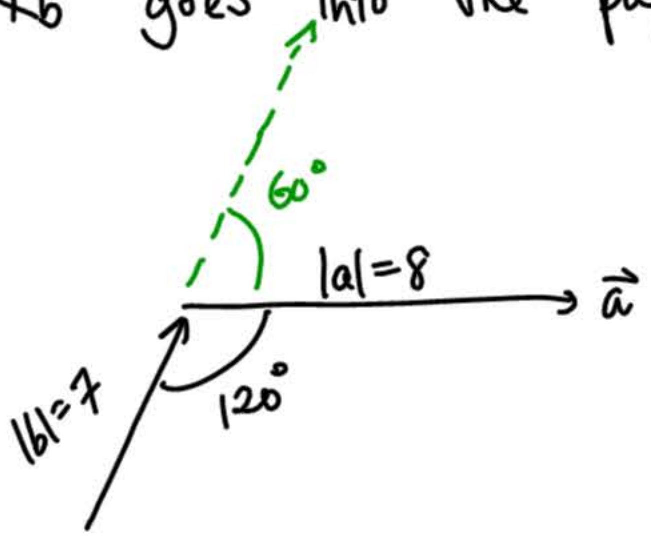
a.



$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$
$$= (5)(4)\sin(45°)$$
$$= 20\frac{\sqrt{2}}{2} = 10\sqrt{2}$$

$\vec{a} \times \vec{b}$ goes into the page.

b.



$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$
$$= (8)(7)\sin(60)$$
$$= 56\frac{\sqrt{3}}{2} = 28\sqrt{3}$$

$\vec{a} \times \vec{b}$ goes out of the page.

A vector $\vec{a}$ is a unit vector if $|\vec{a}| = 1$.

$\vec{a} = \langle 1, -1, 1 \rangle$

$$|\vec{a}| = \sqrt{1^2 + (-1)^2 + 1^2}$$
$$= \sqrt{1+1+1} = \sqrt{3} \neq 1$$