Test 2 Review

October 23, 2018

1. Evaluate the following limits, if they exist.

   (a) $\lim_{x \to 4} \frac{16 - x^2}{x^3 + 64} = \frac{16 - 16}{64 + 64} = 0$

   (b) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4}$

   (c) $\lim_{x \to 5^-} \frac{|x - 5|}{x - 5}$

   Recall: $|x - 5| = \begin{cases} x - 5 & x - 5 \geq 0 \\ -(x - 5) & x - 5 < 0 \end{cases}$

   $$= \begin{cases} x - 5 & x \geq 5 \\ -(x - 5) & x < 5 \end{cases}$$

   $$\lim_{x \to 5^-} \frac{|x - 5|}{x - 5} = \lim_{x \to 5^-} \frac{-(x - 5)}{x - 5} = \lim_{x \to 5^-} -1 = -1$$

   (d) $\lim_{x \to \infty} \frac{x + 3x^2}{4x - 1}$

   $$= \lim_{x \to \infty} \frac{x + 3x^2}{4x - 1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{1 + 3x}{4 - \frac{1}{x}} = \frac{\infty}{4} = \infty$$

   Since infinity is not a finite number, the limit does not exist.

2. Is the following function continuous everywhere?

$$f(x) = \begin{cases} \frac{x^2 + 8x + 12}{|x + 6|} & x \neq -6 \\ 2 & x = -6 \end{cases}$$

$$|x + 6| = \begin{cases} x + 6 & x + 6 \geq 0 \\ -(x + 6) & x + 6 < 0 \end{cases}$$

$$= \begin{cases} x + 6 & x \geq -6 \\ -(x + 6) & x < -6 \end{cases}$$

A good place to check whether the function is continuous is $x = -6$. For $f(x)$ to be continuous, we desire:

$$\lim_{x \to -6^-} f(x) = \lim_{x \to -6^+} f(x) = L = f(-6)$$

$$\lim_{x \to -6^-} f(x) = \lim_{x \to -6^-} \frac{x^2 + 8x + 12}{|x + 6|} = \lim_{x \to -6^-} \frac{(x + 6)(x + 2)}{-(x + 6)} = \lim_{x \to -6^-} -(x + 2) = -(-4) = 4$$

$$\lim_{x \to -6^+} f(x) = \lim_{x \to -6^+} \frac{x^2 + 8x + 12}{|x + 6|} = \lim_{x \to -6^+} \frac{(x + 6)(x + 2)}{x + 6} = \lim_{x \to -6^+} (x + 2) = -4$$

Since the left limit and right limits are not the same, we fail to be continuous at $x = -6$. As such, we cannot be continuous everywhere.

3. What value of c would make the following function continuous everywhere?

$$f(x) = \begin{cases} \dfrac{(x + 2)^3}{|x + 2|} & x \neq -2 \\ c & x = -2 \end{cases}$$

$$|x + 2| = \begin{cases} x + 2 & x + 2 \geq 0 \\ -(x + 2) & x + 2 < 0 \end{cases}$$

$$= \begin{cases} x + 2 & x \geq -2 \\ -(x + 2) & x < -2 \end{cases}$$

We should first check that our function is continuous at $x = -2$. Once again, we desire:

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x) = L = f(-2)$$

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{(x + 2)^3}{|x + 2|} = \lim_{x \to -2^-} \frac{(x + 2)^3}{-(x + 2)} = \lim_{x \to -2^-} -(x + 2)^2 = 0$$

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \frac{(x + 2)^3}{|x + 2|} = \lim_{x \to -2^+} \frac{(x + 2)^3}{x + 2} = \lim_{x \to -2^+} (x + 2)^2 = 0$$

The left and right limits are equal to a common value of 0. To finish up, we need $f(-2)$ also equal to this common value of 0. Therefore, we want $f(-2) = c = 0$. I claim that this function is continuous everywhere else (convince yourself that this is true).

4. Does the following function have any horizontal asymptotes?

$$f(x) = \frac{\sqrt{1+4x^6}}{2-x^3}$$

We note that for $x > 0$, $x^3 = \sqrt{x^6}$ and for $x < 0$, $x^3 = -\sqrt{x^6}$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} &= \lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{\frac{\sqrt{1+4x^6}}{\sqrt{x^6}}}{\frac{2}{x^3}-1} \\
&= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1} \\
&= \frac{\sqrt{0+4}}{0-1} = \frac{2}{-1} = -2
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3} &= \lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\
&= \lim_{x\to-\infty} \frac{-\frac{\sqrt{1+4x^6}}{\sqrt{x^6}}}{\frac{2}{x^3}-1} \\
&= \lim_{x\to-\infty} \frac{-\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1} \\
&= \frac{-\sqrt{0+4}}{0-1} = \frac{-2}{-1} = 2
\end{aligned}$$

In conclusion, we have a horizontal asymptote at $y = -2$ as $x \to +\infty$ and a horizontal asymptote at $y = 2$ as $x \to -\infty$.

5. Differentiate the following by first-principles.

(a) $f(x) = \pi x^{-2}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{\pi}{(x+h)^2} - \frac{\pi}{x^2}}{h} = \pi \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(x+h)^2} - \frac{1}{x^2}\right)$$

$$= \pi \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(x+h)^2}\cdot\frac{x^2}{x^2} - \frac{1}{x^2}\cdot\frac{(x+h)^2}{(x+h)^2}\right)$$

$$= \pi \lim_{h\to 0} \frac{1}{h}\left(\frac{x^2 - (x+h)^2}{x^2(x+h)^2}\right)$$

$$= \pi \lim_{h\to 0} \frac{1}{h}\left(\frac{(x-(x+h))(x+(x+h))}{x^2(x+h)^2}\right)$$

$$= \pi \lim_{h\to 0} \frac{1}{h}\left(\frac{(-h)(2x+h)}{x^2(x+h)^2}\right)$$

$$= \pi \lim_{h\to 0} \frac{-(2x+h)}{x^2(x+h)^2}$$

$$= \pi \cdot \frac{-2x}{x^2\cdot x^2} = \frac{-2\pi}{x^3}$$

$$= -2\pi x^{-3}$$

(b) $g(x) = \sqrt{9-2x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{9-2(x+h)} - \sqrt{9-2x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{9-2(x+h)} - \sqrt{9-2x}}{h} \cdot \frac{(\sqrt{9-2(x+h)} + \sqrt{9-2x})}{(\sqrt{9-2(x+h)} + \sqrt{9-2x})}$$

$$= \lim_{h\to 0} \frac{(9-2(x+h)) - (9-2x)}{h\left(\sqrt{9-2(x+h)} + \sqrt{9-2x}\right)}$$

$$= \lim_{h\to 0} \frac{9-2x-2h-9+2x}{h\left(\sqrt{9-2(x+h)} + \sqrt{9-2x}\right)}$$

$$= \lim_{h\to 0} \frac{-2}{\sqrt{9-2(x+h)} + \sqrt{9-2x}}$$

$$= \frac{-2}{\sqrt{9-2x} + \sqrt{9-2x}} = \frac{-2}{2\sqrt{9-2x}}$$

$$= \frac{-1}{\sqrt{9-2x}}$$

$$= -(9-2x)^{-\frac{1}{2}}$$

6. Differentiate the following functions.

(a) $f(x) = e^{7.3}$

$f'(x) = 0$

(b) $H(u) = (3u-1)(u+2) = 3u^2 + 6u - u - 2 = 3u^2 + 5u - 2$

$H'(u) = 6u + 5$

(c) $R(a) = (3a+1)^2 = 9a^2 + 6a + 1$

$R'(a) = 18a + 6$

(d) $y = \dfrac{x^2+4x+3}{\sqrt{x}} = x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}} = \frac{3\sqrt{x}}{2} + \frac{2}{\sqrt{x}} - \frac{3}{2\sqrt{x^3}}$$

(e) $G(t) = \sqrt{5t} + \dfrac{\sqrt{7}}{t} = \sqrt{5}t^{\frac{1}{2}} + \sqrt{7}t^{-1}$

$$\frac{dG}{dt} = \frac{\sqrt{5}}{2}t^{-\frac{1}{2}} - \sqrt{7}t^{-2} = \frac{\sqrt{5}}{2\sqrt{t}} - \frac{\sqrt{7}}{t^2}$$

(f) $k(r) = e^r + r^e$

$$\frac{dk}{dr} = e^r + er^{e-1}$$

(g) $z = \dfrac{A}{y^{10}} + Be^y = Ay^{-10} + Be^y$

$$\frac{dz}{dy} = -10Ay^{-11} + Be^y = \frac{-10A}{y^{11}} + Be^y$$

(h) $y = e^{x+1} + 1 = (e^x \cdot e^1) + 1$

$y' = e^x \cdot e^1 = e^{x+1}$

(i) $h(r) = \dfrac{ae^r}{b+e^r}$

$$\frac{dh}{dr} = \frac{(ae^r)(b+e^r) - (ae^r)(e^r)}{(b+e^r)^2} = \frac{abe^r + ae^{2r} - ae^{2r}}{(b+e^r)^2} = \frac{abe^r}{(b+e^r)^2}$$

(j) $y = \dfrac{s-\sqrt{s}}{s^2} = s^{-1} - s^{-\frac{3}{2}}$

$$y' = -s^{-2} + \frac{3}{2}s^{-\frac{5}{2}} = -\frac{1}{s^2} + \frac{3}{2\sqrt{s^5}}$$

(k) $y = (z^2 + e^z)\sqrt{z}$

$$\frac{dy}{dz} = (2z + e^z)\sqrt{z} + (z^2 + e^z)\frac{1}{2\sqrt{z}}$$

$$= 2z^{\frac{3}{2}} + \sqrt{z}e^z + \frac{1}{2}z^{\frac{3}{2}} + \frac{e^z}{2\sqrt{z}}$$

$$= \frac{5\sqrt{z^3}}{2} + e^z\left(\sqrt{z} + \frac{1}{2\sqrt{z}}\right)$$

(l) $V(t) = \dfrac{4+t}{te^t}$

$(te^t)' = (1)(e^t) + (t)(e^t) = (t+1)e^t$

$$V'(t) = \frac{(1)(te^t) - (4+t)(te^t)'}{(te^t)^2}$$

$$= \frac{te^t - (4+t)(t+1)e^t}{t^2e^{2t}}$$

$$= \frac{e^t(t - (4+t)(t+1))}{t^2e^{2t}}$$

$$= \frac{(t - 4t - 4 - t^2 - t)}{t^2e^t} = \frac{-(t+2)^2}{t^2e^t}$$

7. Find the equation of the tangent that passes through each function at the given point.

(a) $y = x + \dfrac{2}{x} = x + 2x^{-1}, \qquad P(2,3)$

$y' = 1 - 2x^{-2}$

$y = 3, \quad x = 2, \quad m = y'(2) = 1 - 2(2^{-2}) = 1 - \dfrac{1}{2} = \dfrac{1}{2}$. From $y = mx + b$:

$3 = \frac{1}{2}(2) + b \Longrightarrow b = 2$

The equation of the tangent is $y = \dfrac{1}{2}x + 2$.

(b) $y = \sqrt[4]{x} - x = x^{\frac{1}{4}} - x, \qquad P(1,0)$

$y' = \dfrac{1}{4}x^{-\frac{3}{4}} - 1$

$y = 0, \quad x = 1, \quad m = \dfrac{1}{4}(1)^{-\frac{3}{4}} - 1 = \dfrac{1}{4} - 1 = -\dfrac{3}{4}$. From $y = mx + b$:

$0 = -\dfrac{3}{4}(1) + b \Longrightarrow b = \dfrac{3}{4}$

The equation of the tangent is $y = -\dfrac{3}{4}x + \dfrac{3}{4}$.

8. A line intersects the curve $f(x) = -3x^3 + 2x + 1$ at the points $(-1, 2)$ and $(1, 0)$. For what values of x would the tangent of $f(x)$ be parallel to this line?

Using the given points, the slope of the given line is:

$$m = \frac{\Delta y}{\Delta x} = \frac{0-2}{1-(-1)} = \frac{-2}{2} = -1$$

We need to look for x values such that the tangent of $f(x)$ has slope equal to -1. In other words, we need to find x such that $f'(x) = -1$.

$f'(x) = -9x^2 + 2$.

$$-9x^2 + 2 = -1$$

$$-9x^2 + 3 = 0$$

$$x^2 - \frac{1}{3} = 0$$

$$\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right) = 0$$

$$x_1 = \frac{1}{\sqrt{3}}, \quad x_2 = \frac{-1}{\sqrt{3}}$$

We conclude that the tangent of $f(x)$ will be parallel to the given line when $x = x_1$ and when $x = x_2$.

9. Find a formula to describe the instantaneous rate of change for $g(t) = \frac{t}{e^t}$. Does $g(t)$ have any horizontal tangents?

$$g'(t) = \frac{(1)(e^t) - t(e^t)}{(e^t)^2}$$

$$= \frac{e^t(1-t)}{(e^t)^2}$$

$$= \frac{1-t}{e^t}$$

This is the instantaneous rate of change.

To find whether there are any horizontal tangents, we set $g'(t) = 0$.

$$g'(t) = 0$$

$$\frac{1-t}{e^t} \cdot e^t = 0 \cdot e^t \qquad \text{(We can cancel } e^t \text{ because it is never 0)}$$

$$1 - t = 0$$

$$t = 1$$

$g(t)$ has a horizontal tangent when $t = 1$.