Tutorial 11

Week of November 26, 2018

1. Suppose we have two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ in $\mathbb{R}^3$. Show that for any non-zero scalars $c, d \in \mathbb{R}$:

   (a) $(c\mathbf{a}) \cdot (d\mathbf{b}) = cd\,(\mathbf{a} \cdot \mathbf{b})$

   $$\begin{aligned}(c\mathbf{a}) \cdot (d\mathbf{b}) &= \langle ca_1, ca_2, ca_3 \rangle \cdot \langle db_1, db_2, db_3 \rangle \\ &= cda_1b_1 + cda_2b_2 + cda_3b_3 \\ &= cd(a_1b_1 + a_2b_2 + a_3b_3) \\ &= cd(\mathbf{a} \cdot \mathbf{b})\end{aligned}$$

   (b) $(c\mathbf{a}) \times (d\mathbf{b}) = cd\,(\mathbf{a} \times \mathbf{b})$

   $$\begin{aligned}(c\mathbf{a}) \times (d\mathbf{b}) &= \langle ca_1, ca_2, ca_3 \rangle \times \langle db_1, db_2, db_3 \rangle \\ &= \langle ca_2db_3 - db_2ca_3, \quad ca_3db_1 - db_3ca_1, \quad ca_1db_2 - db_1ca_2 \rangle \\ &= \langle cd(a_2b_3 - b_2a_3), \quad cd(a_3b_1 - b_3a_1), \quad cd(a_1b_2 - b_1a_2) \rangle \\ &= cd\langle a_2b_3 - b_2a_3, \quad a_3b_1 - b_3a_1, \quad a_1b_2 - b_1a_2 \rangle \\ &= cd(\mathbf{a} \times \mathbf{b})\end{aligned}$$

2. Find $\mathbf{a} \cdot \mathbf{b}$.

   (a) $\mathbf{a} = \langle 5, -2 \rangle, \quad \mathbf{b} = \langle 3, 4 \rangle$

   $\langle 5, -2 \rangle \cdot \langle 3, 4 \rangle = (5)(3) + (-2)(4) = 15 - 8 = 7$

   (b) $\mathbf{a} = \langle 6, -2, 3 \rangle, \quad \mathbf{b} = \langle 2, 5, -1 \rangle$

   $\langle 6, -2, 3 \rangle \cdot \langle 2, 5, -1 \rangle = (6)(2) + (-2)(5) + (3)(-1) = 12 - 10 - 3 = -1$

   (c) $\mathbf{a} = 2\mathbf{i} + \mathbf{j}, \quad \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

   $(2\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} - \mathbf{j} + \mathbf{k}) = (2)(1) + (1)(-1) + (0)(1) = 2 - 1 + 0 = 1$

   (d) $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{b} = 4\mathbf{i} + 5\mathbf{k}$

   $(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (4\mathbf{i} + 5\mathbf{k}) = (3)(4) + (2)(0) + (-1)(5) = 12 + 0 - 5 = 7$

   (e) $|\mathbf{a}| = 7, \quad |\mathbf{b}| = 4, \quad \theta = 30°$

   $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = (7)(4)\cos 30 = 28\frac{\sqrt{3}}{2} = 14\sqrt{3}$

3. Find $\mathbf{a} \times \mathbf{b}$. Verify that the cross product is orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

(a) $\mathbf{a} = \langle 4, 3, -2 \rangle, \quad \mathbf{b} = \langle 2, -1, 1 \rangle$

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \langle 3-2, -4-4, -4-6 \rangle \\ &= \langle 1, -8, -10 \rangle := \mathbf{c} \\ \mathbf{a} \cdot \mathbf{c} &= \langle 4, 3, -2 \rangle \cdot \langle 1, -8, -10 \rangle \\ &= 4 - 24 + 20 = 0 \\ \mathbf{b} \cdot \mathbf{c} &= \langle 2, -1, 1 \rangle \cdot \langle 1, -8, -10 \rangle \\ &= 2 + 8 - 10 = 0 \end{aligned}$$

Since $\mathbf{a} \cdot \mathbf{c} = 0$, $\mathbf{a}$ is orthogonal to $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. The same can be said of $\mathbf{b}$. This was expected since the cross product between two vectors finds a third vector that is orthogonal to the original two.

(b) $\mathbf{a} = 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$

$$\begin{aligned} \mathbf{a} &= 3(\mathbf{i} + \mathbf{j} - \mathbf{k}) = 3\langle 1, 1, -1 \rangle \\ \mathbf{b} &= 3(\mathbf{i} + \mathbf{j} - \mathbf{k}) = 3\langle 1, -1, 1 \rangle \end{aligned}$$

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= 3\langle 1, 1, -1 \rangle \times 3\langle 1, -1, 1 \rangle = 9\left(\langle 1, 1, -1 \rangle \times \langle 1, -1, 1 \rangle\right) \\ &= 9\langle 1-1, -1-1, -1-1 \rangle \\ &= 9\langle 0, -2, -2 \rangle := \mathbf{c} \\ \mathbf{a} \cdot \mathbf{c} &= 3\langle 1, 1, -1 \rangle \cdot 9\langle 0, -2, -2 \rangle \\ &= 27\left(\langle 1, 1, -1 \rangle \cdot \langle 0, -2, -2 \rangle\right) \\ &= 27(0 - 2 + 2) = 0 \\ \mathbf{b} \cdot \mathbf{c} &= 3\langle 1, -1, 1 \rangle \cdot 9\langle 0, -2, -2 \rangle \\ &= 27\left(\langle 1, -1, 1 \rangle \cdot \langle 0, -2, -2 \rangle\right) \\ &= 27(0 + 2 - 2) = 0 \end{aligned}$$

4. Find the angles of the triangle given by the points $P(2, 0)$, $Q(0, 3)$, $R(3, 4)$.

$$\overrightarrow{PQ} = \langle 0-2, 3-0 \rangle = \langle -2, 3 \rangle, \quad |\overrightarrow{PQ}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

$$\overrightarrow{PR} = \langle 3-2, 4-0 \rangle = \langle 1, 4 \rangle, \quad |\overrightarrow{PR}| = \sqrt{(1)^2 + (4)^2} = \sqrt{17}$$

$$\begin{aligned} \overrightarrow{PQ} \cdot \overrightarrow{PR} &= \langle -2, 3 \rangle \cdot \langle 1, 4 \rangle = (-2)(1) + (3)(4) = -2 + 12 = 10 \\ &= |\overrightarrow{PQ}||\overrightarrow{PR}| \cos\theta_p = \sqrt{13}\sqrt{17} \cos\theta_p = \sqrt{221} \cos\theta_p \end{aligned}$$

$$\sqrt{221} \cos\theta_p = 10$$

$$\theta_p = \arccos\left(\frac{10}{\sqrt{221}}\right) \approx 47.7^\circ$$

Using vectors $\overrightarrow{QP}$ (which is just negative $\overrightarrow{PQ}$) and $\overrightarrow{QR}$ we find that $\theta_q \approx 74.7^\circ$. Since the angles of a triangle sum to 180°, $\theta_r = 180 - 47.7 - 74.7 = 57.6^\circ$.

5. Using the properties of the cross product, compute the following:

(a) $\mathbf{k} \times (\mathbf{i} - 2\mathbf{j})$

$$
\begin{aligned}
&= \mathbf{k} \times (\mathbf{i} + (-2)\mathbf{j}) \\
&= (\mathbf{k} \times \mathbf{i}) + (\mathbf{k} \times (-2)\mathbf{j}) \\
&= (\mathbf{k} \times \mathbf{i}) - 2(\mathbf{k} \times \mathbf{j}) \\
&= \mathbf{j} - 2(-\mathbf{i}) \\
&= \mathbf{j} + 2\mathbf{i}
\end{aligned}
$$

(b) $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j})$

$$
\begin{aligned}
&= (\mathbf{i} \times (\mathbf{i} - \mathbf{j})) + (\mathbf{j} \times (\mathbf{i} - \mathbf{j})) \\
&= (\mathbf{i} \times \mathbf{i}) - (\mathbf{i} \times \mathbf{j}) + (\mathbf{j} \times \mathbf{i}) - (\mathbf{j} \times \mathbf{j}) \\
&= 0 - \mathbf{k} - \mathbf{k} - 0 \\
&= -2\mathbf{k}
\end{aligned}
$$