Tutorial 12

Week of December 3, 2018

1. Find parametric and symmetric equations of the following lines.

   (a) The line that passes through the points $P_1(-8, 1, 4)$ and $P_2(3, -2, 4)$.

   $\overrightarrow{P_1P_2} = \langle 3+8, -2-1, 4-4\rangle = \langle 11, -3, 0\rangle$
   $\overrightarrow{OP_1} = \langle -8, 1, 4\rangle$

   $$\begin{aligned}\mathbf{r} &= \mathbf{r}_0 + t\mathbf{d}\\ &= \overrightarrow{P_1} + t\overrightarrow{P_1P_2}\\ &= \langle -8, 1, 4\rangle + t\langle 11, -3, 0\rangle, \quad t \in \mathbb{R}\end{aligned}$$

   The parametric equations are $x = -8 + 11t$, $y = 1 - 3t$, $z = 4 + 0t = 4$.
   The symmetric equations are $\frac{x+8}{11} = \frac{1-y}{3}$, $z = 4$.

   (b) The line that passes through $P(2, 1, 0)$ and is perpendicular to both $\mathbf{v} = \mathbf{i} + \mathbf{j}$ and $\mathbf{w} = \mathbf{j} + \mathbf{k}$.

   We seek a line with a direction vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$. This vector will be $\mathbf{v} \times \mathbf{w}$.
   $\mathbf{v} \times \mathbf{w} = \langle 1-0, 0-1, 1-0\rangle = \langle 1, -1, 1\rangle$.
   $\overrightarrow{OP} = \langle 2, 1, 0\rangle$.

   $$\begin{aligned}\mathbf{r} &= \mathbf{r}_0 + t\mathbf{d}\\ &= \overrightarrow{OP} + t(\mathbf{v} \times \mathbf{w})\\ &= \langle 2, 1, 0\rangle + t\langle 1, -1, 1\rangle, \quad t \in \mathbb{R}\end{aligned}$$

   The parametric equations are $x = 2 + t$, $y = 1 - t$, $z = 0 + t = t$.
   The symmetric equations are $x - 2 = 1 - y = z$.

   (c) The line that passes through $P(1, 0, 6)$ and is perpendicular to the plane $x + 3y + z = 5$.

   The equation of the given plane tells us that is has normal vector $\mathbf{n} = \langle 1, 3, 1\rangle$. For a line to be perpendicular to a plane means that it has a direction vector that is parallel to the normal vector of the plane. So we take $\mathbf{n}$ as the direction vector of the line (or any scalar multiple of it).
   Construct $\overrightarrow{OP} = \langle 1, 0, 6\rangle$.

   $$\begin{aligned}\mathbf{r} &= \mathbf{r}_0 + t\mathbf{d}\\ &= \overrightarrow{OP} + t\mathbf{n}\\ &= \langle 1, 0, 6\rangle + t\langle 1, 3, 1\rangle, \quad t \in \mathbb{R}\end{aligned}$$

   The parametric equations are $x = 1 + t$, $y = 0 + 3t = 3t$, $z = 6 + t$.
   The symmetric equations are $x - 1 = \frac{1}{3}y = z - 6$.

2. Find the scalar equation of the following planes.

(a) The plane that contains the point $(3, -2, 8)$ and is parallel to the plane $z = x + y$.

Rearranging $z = x + y$ we get $x + y - z = 0$. The normal vector of this plane is $\mathbf{n} = \langle 1, 1, -1 \rangle$. $\mathbf{r}_0 = \langle 3, -2, 8 \rangle$.

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) &= 0 \\ \langle 1, 1, -1 \rangle \cdot \langle x - 3, y + 2, z - 8 \rangle &= 0 \\ (1)(x - 3) + (1)(y + 2) + (-1)(z - 8) &= 0 \\ x + y + z + 7 &= 0 \end{aligned}$$

(b) The plane that contains the line $x = 1 + t$, $y = 2 - t$, $z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$.

From the given symmetric equations and equation of the plane, we can extract $\mathbf{r}_0 = \langle 1, 2, 4 \rangle$ and $\mathbf{n} = \langle 5, 2, 1 \rangle$, respectively. The plane we seek is parallel to the given plane so its normal vector can be any scalar multiple of $\mathbf{n}$. For simplicity just take $\mathbf{n}$ as is. Since we want a plane that contains the given line, the direction vector of the line should be perpendicular to the normal vector of the plane we seek. Once again:

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) &= 0 \\ \langle 5, 2, 1 \rangle \cdot \langle x - 1, y - 2, z - 4 \rangle &= 0 \\ (5)(x - 1) + (2)(y - 2) + (1)(z - 4) &= 0 \\ 5x + 2y + z - 13 &= 0 \end{aligned}$$

3. Determine whether the following statements are true or false.

(a) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a} \in \mathbb{R}^3$.

This is true from properties of the cross product. Define $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$. Then:

$$\begin{aligned} \mathbf{a} \times \mathbf{a} &= \langle a_1, a_2, a_3 \rangle \times \langle a_1, a_2, a_3 \rangle \\ &= \langle a_2 a_3 - a_2 a_3, a_3 a_1 - a_3 a_1, a_1 a_2 - a_1 a_2 \rangle \\ &= \langle 0, 0, 0 \rangle \\ &= \mathbf{0} \end{aligned}$$

(b) The line through $P(-2, 4, 0)$ and $Q(1, 1, 1)$ is perpendicular to the line through $R(2, 3, 4)$ and $S(3, -1, -8)$.

A direction vector passing through P and Q is $\overrightarrow{PQ} = \langle 1 + 2, 1 - 4, 1 - 0 \rangle = \langle 3, -3, 1 \rangle$.
A direction vector passing through R and S is $\overrightarrow{RS} = \langle 3 - 2, -1 - 3, -8 - 4 \rangle = \langle 1, -4, -12 \rangle$.
If these two lines are perpendicular, the dot product of the direction vectors will be 0.

$$\begin{aligned} \overrightarrow{PQ} \cdot \overrightarrow{RS} &= \langle 3, -3, 1 \rangle \cdot \langle 1, -4, -12 \rangle \\ &= (3)(1) + (-3)(-4) + (1)(-12) \\ &= 3 + 12 - 12 \\ &= 3 \end{aligned}$$

Since the dot product is not zero, the two direction vectors are not perpendicular so the lines are not perpendicular. So this statement is false.

(c) The vector $\mathbf{a} = \langle 3, -1, 2 \rangle$ is parallel to the plane $6x - 2y + 4z = 1$.

Since the normal vector of a plane is perpendicular to its surface, if a vector is parallel to the plane, it must be perpendicular to the plane's normal vector. The normal vector of the given plane is $\mathbf{n} = \langle 6, -2, 4 \rangle$. So we want to check if $\langle 3, -1, 2 \rangle \cdot \mathbf{n} = 0$.

$$\begin{aligned}\langle 3, -1, 2 \rangle \cdot \mathbf{n} &= \langle 3, -1, 2 \rangle \cdot \langle 6, -2, 4 \rangle \\ &= (3)(6) + (-1)(-2) + (2)(4) \\ &= 18 + 2 + 8 \\ &= 28\end{aligned}$$

Since the dot product is not zero, the given vector is not perpendicular to the plane's normal, meaning that the vector is also not parallel to the given plane. In fact, without doing any calculations, we could have predicted this since the given vector is a scalar multiple of the plane's normal vector, implying that the given vector is in fact parallel to the normal vector of the plane, meaning that the vector is actually perpendicular to the given plane. We conclude that the given statement is false.