Tutorial 3

Week of September 24, 2018

1. Suppose $g(x) = 3 + x + e^x$. Find $g^{-1}(4)$ (without finding an explicit formula).

   $g^{-1}(4)$ is telling us that the y-value is 4, and we are looking for the x-value associated to that.

   $$\begin{aligned} 4 &= 3 + x + e^x \\ 1 &= x + e^x \end{aligned}$$

   The value of x that satisfies the above is $x = 0$. Therefore, $g^{-1}(4) = 0$.

2. Find a formula for the inverse.

   (a) $y = 1 + \sqrt{2 + 3x}$

   $$\begin{aligned} x &= 1 + \sqrt{2 + 3y} \\ x - 1 &= \sqrt{2 + 3y} \\ \frac{(x-1)^2 - 2}{3} &= y \end{aligned}$$

   Notice that in finding the inverse of a square root function, we obtain an equation for a parabola in vertex form. But is it really a parabola? [Hint: Look at domain and range]

   (b) $y = e^{2x-1}$

   $$\begin{aligned} x &= e^{2y-1} \\ \ln(x) &= 2y - 1 \\ \frac{\ln(x) + 1}{2} &= y \end{aligned}$$

   (c) $y = \ln(x + 3)$

   $$\begin{aligned} x &= \ln(y + 3) \\ e^x &= y + 3 \\ e^x - 3 &= y \end{aligned}$$

(d) $y = \dfrac{1 - e^{-x}}{1 + e^{-x}}$

$$\begin{aligned} x &= \frac{1 - e^{-y}}{1 + e^{-y}} \\ x(1 + e^{-y}) &= 1 - e^{-y} \\ x + xe^{-y} &= 1 - e^{-y} \\ xe^{-y} + e^{-y} &= 1 - x \\ e^{-y}(x + 1) &= 1 - x \\ e^{-y} &= \frac{1 - x}{x + 1} \\ -y &= \ln\left(\frac{1 - x}{x + 1}\right) \\ y &= -\ln\left(\frac{1 - x}{x + 1}\right) \\ &= \ln\left(\frac{x + 1}{1 - x}\right) \end{aligned}$$

3. Evaluate the following.

(a) $\log_2(32)$

$2^5 = 32$. Therefore $\log_2(32) = 5$.

(b) $\log_8(2)$

$8^{1/3} = 2$. Therefore $\log_8(2) = 1/3$.

(c) $\log_{10}(40) + \log_{10}(2.5)$

$= \log_{10}(40 \cdot 2.5) = \log_{10}(100) = 2$

(d) $\log_8(60) - \log_8(3) - \log_8(5)$

$= \log_8(60/3) - \log_8(5)$
$= \log_8(20) - \log_8(5)$
$= \log_8(20/5)$
$= \log_8(4)$
$= 2/3$

(e) $e^{-\ln(2)}$

$= e^{\ln(2^{-1})} = 2^{-1} = 1/2$

(f) $e^{\ln(\ln(e^3))}$

$= \ln(e^3) = 3\ln(e) = 3$

4. Solve the following.

   (a) $2^{x-5} = 3$

$$\begin{aligned} \ln(2^{x-5}) &= \ln(3) \\ (x-5)\ln(2) &= \ln(3) \\ x &= \frac{\ln(3)}{\ln(2)} + 5 \end{aligned}$$

   OR

$$\begin{aligned} \log_2(2^{x-5}) &= \log_2(3) \\ (x-5)\log_2(2) &= \log_2(3) \\ x &= \log_2(3) + 5 \end{aligned}$$

   (b) $\ln(x) + \ln(x-1) = 1$

$$\begin{aligned} \ln(x(x-1)) &= 1 \\ x(x-1) &= e^1 \\ x^2 - x - e^1 &= 0 \end{aligned}$$

   We then use the quadratic formula. One answer will be negative and one will be positive. Only the positive solution is valid since in our original equation, we needed $x > 0$ and $x - 1 > 0$, implying that $x > 1$.

   (c) $\ln(\ln(x)) = 1$

$$\begin{aligned} e^{\ln(\ln(x))} &= e^1 \\ \ln(x) &= e^1 \\ x &= e^{e^1} \end{aligned}$$

   (d) $e^{ax} = Ce^{bx}$

$$\frac{e^{ax}}{e^{bx}} = C$$

$$\begin{aligned} e^{x(a-b)} &= C \\ x(a-b) &= \ln(C) \\ x &= \frac{\ln(C)}{a-b} \end{aligned}$$

5. Explain what it means to say:

$$\lim_{x \to 1^-} f(x) = 3 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 7$$

   In this situation, can the limit of $f(x)$ as $x \to 1$ exist?

   The first limit says that as x approaches 1 from the left, $f(x)$ approaches a value of 3. The limit on the right says that as x approaches 1 from the right, $f(x)$ approaches a value of 7. Since the left limit and right limits are not the same we say that the limit of $f(x)$ as x approaches 1 does not exist.

6. State the following limits.

   (a) $\lim_{x \to 5^-} g(x) = 2$

   (b) $\lim_{x \to 5^+} g(x) = 2$

   (c) $\lim_{x \to 5} g(x) = 2$. Since the left and right limits were equal, this limit must also be the same. However, notice that despite these limits equalling 2, $f(x)$ does not actually take on the value of 2 when x is 5!!!

7. Determine the infinite limit.

   (a) $\lim_{x \to 5^-} \frac{x+1}{x-5}$

   Consider x values less than 5 such as 4.99, 4.999, etc. For each of these values that we plug in, the numerator will be positive while the denominator will be negative and small in magnitude (dividing by such a small number makes our function grow very fast). Therefore we are approaching negative infinity.

   (b) $\lim_{x \to 3^-} \frac{\sqrt{x}}{(x-3)^5}$

   Consider x values less than 3 such as 2.99, 2.999, etc. For each of these values, the numerator will be positive. The denominator will be a negative number raised to the power of 5, which is still negative. The denominator is very, very small in magnitude so our function will be approaching negative infinity.

   (c) $\lim_{x \to 2\pi^-} x \csc(x)$

   We can rewrite our function as $x/\sin(x)$. If we draw out the graph of $\sin(x)$, we notice that our function is a small negative number when we are slightly less than 2π. The numerator of $x/\sin(x)$ is positive while the denominator is a negative number that is small in magnitude so we will be approaching negative infinity.