Tutorial 5

Week of October 15, 2018

Remember to review Figure 7 on Page 158 for instances when a function is not differentiable at a point.

1. Find the equation of the tangent line to the curve at the given point.

$$f(x) = \frac{1}{x-1}, \quad P(2,1)$$

Recall that $\frac{a}{b}$ is equivalent to $a \times \frac{1}{b}$. It follows that:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad = \quad \lim_{h\to 0} \frac{1}{h}\left(f(x+h) - f(x)\right)$$

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(2+h)-1} - \frac{1}{2-1}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{h+1} - \frac{1}{1}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{h+1} - \frac{1}{1}\frac{(h+1)}{(h+1)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1-(h+1)}{h+1}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{h+1}\right) = \lim_{h\to 0} \frac{-1}{h+1}$$

$$= -1$$

$y = 1$, $m = -1$, $x = 2$, $b = ?$

$1 = (-1)(2) + b$
$b = 1 + 2 = 3$

The equation of the tangent line to the curve $f(x)$ at the point $P(2,1)$ is $y = -x + 3$.

2. (a) Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ for any given x in the domain.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{x+h}}\frac{\sqrt{x}}{\sqrt{x}} - \frac{1}{\sqrt{x}}\frac{\sqrt{x+h}}{\sqrt{x+h}}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x(x+h)}}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x(x+h)}}\frac{\left(\sqrt{x} + \sqrt{x+h}\right)}{\left(\sqrt{x} + \sqrt{x+h}\right)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{x - (x+h)}{\sqrt{x(x+h)}\left(\sqrt{x} + \sqrt{x+h}\right)}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{\sqrt{x(x+h)}\left(\sqrt{x} + \sqrt{x+h}\right)}\right)$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x(x+h)}\left(\sqrt{x} + \sqrt{x+h}\right)}$$

$$= \frac{-1}{\sqrt{x^2}\left(2\sqrt{x}\right)} = \frac{-1}{2\sqrt{x^3}}$$

$$= -\frac{1}{2}x^{-3/2}$$

Given some arbitrary point (v, w) that lies on this curve, the equation of its tangent line will be:

$$y = \left(-\frac{1}{2}v^{-3/2}\right)(x) + b$$

(b) Find equations of the tangent lines at the points $(1, 1)$ and $(4, \frac{1}{2})$.

i) At the point $(1, 1)$

$$1 = -\frac{1}{2}(1)^{-3/2}(1) + b$$

$$1 = -\frac{1}{2} + b$$

$$b = \frac{3}{2}$$

$$y = -\frac{1}{2}x + \frac{3}{2}$$

ii) At the point $\left(4, \frac{1}{2}\right)$

$$\frac{1}{2} = -\frac{1}{2}(4)^{-3/2}(4) + b$$

$$\frac{1}{2} = -\frac{1}{2\sqrt{4}} + b$$

$$\frac{1}{2} = -\frac{1}{4} + b$$

$$b = \frac{3}{4}$$

$$y = -\frac{1}{16}x + \frac{3}{4}$$

3. If an equation of the tangent line to the curve $y = f(x)$ at the point where $a = 2$ is $y = 4x - 5$, find $f(2)$ and $f'(2)$.

Given that the equation of the tangent at $a = 2$ is $y = 4x - 5$, it is immediate that $f'(2) = 4$. The point $(2, f(2))$ lies on both $f(x)$ and the given tangent line. Therefore we can plug $x = 2$ directly into the equation of the tangent to obtain $f(2)$:

$$f(2) = 4(2) - 5 = 3$$

4. If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.

From the fact that $(4, 3)$ is a point on $f(x)$, it is obvious that $f(4) = 3$. Since we are given two points that the tangent line passes through, we can calculate the slope, which is $f'(4)$. The slope of the tangent line at $(4, 3)$ is:

$$f'(4) = m = \frac{\Delta y}{\Delta x} = \frac{3-2}{4-0} = \frac{1}{4}$$

5. Find the derivative of the function using the definition of derivative.

(a) $f(x) = x^{3/2}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^3} - \sqrt{x^3}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^3} - \sqrt{x^3}}{h} \cdot \frac{\left(\sqrt{(x+h)^3} + \sqrt{x^3}\right)}{\left(\sqrt{(x+h)^3} + \sqrt{x^3}\right)} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h\left(\sqrt{(x+h)^3} + \sqrt{x^3}\right)}$$

$$= \lim_{h\to 0} \frac{(x+h-x)\left((x+h)^2 + (x+h)x + x^2\right)}{h\left(\sqrt{(x+h)^3} + \sqrt{x^3}\right)} = \lim_{h\to 0} \frac{h\left((x+h)^2 + (x+h)x + x^2\right)}{h\left(\sqrt{(x+h)^3} + \sqrt{x^3}\right)}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 + (x+h)x + x^2}{\sqrt{(x+h)^3} + \sqrt{x^3}} = \frac{x^2 + x^2 + x^2}{\sqrt{x^3} + \sqrt{x^3}}$$

$$= \frac{3x^2}{2\sqrt{x^3}} = \frac{3}{2} x^{\frac{4}{2} - \frac{3}{2}} = \frac{3}{2} x^{1/2}$$

(b) $f(x) = x^4$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} = \lim_{h\to 0} \frac{\left((x+h)^2 - x^2\right)\left((x+h)^2 + x^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left((x+h-x)(x+h+x)\right)\left((x+h)^2 + x^2\right)}{h} = \lim_{h\to 0} \frac{h(h+2x)\left((x+h)^2 + x^2\right)}{h}$$

$$= \lim_{h\to 0} (h+2x)\left((x+h)^2 + x^2\right)$$

$$= (2x)(x^2 + x^2) = (2x)(2x^2) = 4x^3$$

Something to notice:

$f(x) = x^{3/2}$

$f'(x) = \frac{3}{2}x^{1/2} = \frac{3}{2}x^{((3/2)-1)}$

$f(x) = x^4$

$f'(x) = 4x^3 = 4x^{(4-1)}$

As you should have seen in lecture this week, we can find general formulas for the derivatives of certain functions so that we don't need to go back to first-principles each time.