Tutorial 6

Week of October 22, 2018

1. Differentiate the following functions.

   (a) $f(x) = 2^{40}$

   $f'(x) = 0$ because 2^{40} is a constant.

   (b) $f(x) = e^5$

   While e^x is a function of x, e^5 is a constant. Therefore $f'(x) = 0$.

   (c) $f(x) = (3x^2 - 5x)e^x$

   $$\begin{aligned}\frac{df}{dx} &= (6x-5)e^x + (3x^2-5x)e^x \\ &= 6xe^x - 5e^x + 3x^2e^x - 5xe^x \\ &= xe^x - 5e^x + 3x^2e^x \\ &= e^x(3x^2 + x - 5)\end{aligned}$$

   (d) $g(x) = x^2(1-2x) = x^2 - 2x^3$

   $g'(x) = 2x - 6x^2$

   (e) $y = \sqrt[3]{x}(2+x) = 2x^{\frac{1}{3}} + x^{\frac{4}{3}}$

   $y' = \frac{2}{3}x^{-\frac{2}{3}} + x^{\frac{1}{3}}$

   (f) $g(x) = \dfrac{x^2-2}{2x+1}$

   $$\begin{aligned}\frac{dg}{dx} &= \frac{(2x)(2x+1)-(x^2-2)(2)}{(2x+1)^2} = \frac{4x^2+2x-(2x^2-4)}{4x^2+4x+1} = \frac{4x^2-2x^2+2x+4}{4x^2+4x+1} \\ &= \frac{2x^2+2x+4}{4x^2+4x+1}\end{aligned}$$

   (g) $h(x) = x^{2.4} + e^{2.4}$

   $h'(x) = 2.4x^{1.4}$

(h) $y = e^{x+1} + 1 = (e^x \cdot e^1) + 1$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \cdot e^1) + \frac{d}{dx}(1)$$

$$= e^1 \cdot \frac{d}{dx}e^x + 0$$

$$= e^1 \cdot e^x = e^{x+1}$$

2. Find the equation of the tangent line of $y = 2e^x + x$ at the point $P(0, 2)$.

   $y' = 2e^x + 1$. Then the slope of the tangent line when $x = 0$ is $2e^0 + 1 = 3$.

   $y = mx + b$
   $2 = 3(0) + b$
   $b = 2$

   The equation of the tangent line is $y = 3x + 2$.

3. Find the equation of the tangent line of $y = \dfrac{1+x}{1+e^x}$ at the point $P(0, \frac{1}{2})$.

   $$y' = \frac{(1)(1+e^x) - (1+x)(e^x)}{(1+e^x)^2}$$

   Then the slope of the tangent line when $x = 0$ is: $\dfrac{(1+e^0) - (1+0)(e^0)}{(1+e^0)^2} = \dfrac{2-1}{4} = \dfrac{1}{4}$.

   $y = mx + b$
   $\frac{1}{2} = \frac{1}{4}(0) + b$
   $b = \frac{1}{2}$

   The equation of the tangent line is $y = \frac{1}{4}x + \frac{1}{2}$.

4. Show that $y = 2e^x + 5x^3 + 3x$ does not have a tangent with slope 2.

   $y' = 2e^x + 15x^2 + 3$. Suppose that y had a tangent with slope 2. Then:

   $$2e^x + 15x^2 + 3 = 2 \quad \Longrightarrow \quad 2e^x + 15x^2 = -1$$

   But $2e^x > 0 \ \forall x \in \mathbb{R}$ and $15x^2 \geq 0 \ \forall x \in \mathbb{R}$. Therefore their sum cannot be negative. As such, y does not have a tangent with slope 2.

5. Find the equation of the tangent line to the curve $f(x) = x^4 + 1$ that is parallel to $32x - y = 15$.

   $y' = 4x^3$. Rearranging the equation of the given parallel line for clarity, we have $y = 32x - 15$. If we want the tangent of $f(x)$ to be parallel to the given line, we want them to have the same slope. In other words, we want $f'(x) = 4x^3 = 32$. Solving for x, we get $x = 2$.
   We know that the curve $f(x)$ and its tangent line share the common point of $(2, f(2))$. $f(2) = 2^4 + 1 = 17$. So $(2, 17)$ is the common point.

$$y = mx + b$$
$$17 = 32(2) + b$$
$$b = 17 - 64 = -47$$

The tangent of $f(x)$ parallel to the given line has equation $y = 32x - 47$.

6. Given $h(2) = 4$ and $h'(2) = -3$, find:
$$\frac{d}{dx}\left(\frac{h(x)}{x}\right)\bigg|_{x=2}$$

$$\frac{d}{dx}\left(\frac{h(x)}{x}\right) = \frac{h'(x)\cdot x - h(x)\cdot(1)}{x^2}$$

$$\begin{aligned}\frac{d}{dx}\left(\frac{h(x)}{x}\right)\bigg|_{x=2} &= \frac{h'(2)\cdot(2) - h(2)}{4}\\ &= \frac{(-3)\cdot(2) - 4}{4}\\ &= -\frac{10}{4}\\ &= -\frac{5}{2}\end{aligned}$$