Tutorial 7

Week of October 29, 2018

1. Differentiate the following functions.

   (a) $f(x) = x^2 \sin x$

   $f'(x) = 2x \sin x + x^2 \cos x$

   (b) $g(\theta) = e^\theta(\tan\theta - \theta)$ Recall that $\sec^2 x - 1 = \tan^2 x$

   $$\begin{aligned}\frac{dg}{d\theta} &= e^\theta(\tan\theta - \theta) + e^\theta(\sec^2\theta - 1)\\ &= e^\theta(\tan\theta - \theta + \sec^2\theta - 1)\\ &= e^\theta(\tan^2\theta + \tan\theta - \theta)\end{aligned}$$

   (c) $f(t) = \dfrac{\cot t}{e^t} = \dfrac{\cos t}{e^t \sin t}$

   $$\begin{aligned}f'(t) &= \frac{-\sin t(e^t \sin t) - \cos t(e^t \sin t + e^t \cos t)}{(e^t \sin t)^2}\\ &= \frac{-e^t \sin^2 t - e^t \cos^2 t - e^t \sin t \cos t}{(e^t \sin t)^2}\\ &= \frac{-e^t(\sin^2 t + \cos^2 t) - e^t \sin t \cos t}{(e^t \sin t)^2}\\ &= \frac{-e^t(1 + \sin t \cos t)}{e^{2t} \sin^2 t}\\ &= \frac{-(1 + \sin t \cos t)}{e^t \sin^2 t}\end{aligned}$$

   (d) $r(\theta) = \sin\theta\cos\theta$

   $$\begin{aligned}r'(\theta) &= (\cos\theta)(\cos\theta) + (\sin\theta)(-\sin\theta)\\ &= \cos^2\theta - \sin^2\theta\\ &= \cos 2x\end{aligned}$$

   (e) $k(x) = \sin^2 x = (\sin x)^2$

   $$\begin{aligned}k'(x) &= 2\sin x \cos x\\ &= \sin 2x\end{aligned}$$

(f) $f(x) = (5x^6 + 2x^3)^4$

$f'(x) = 4(5x^6 + 2x^3)^3(30x^5 + 6x^2)$

(g) $g(x) = \dfrac{1}{\sqrt[3]{x^2 - 1}} = (x^2 - 1)^{-\frac{1}{3}}$

$g'(x) = -\dfrac{1}{3}(x^2 - 1)^{-\frac{4}{3}}(2x)$

(h) $h(x) = e^{x^2 - x}$

$h'(x) = e^{x^2 - x}(2x - 1)$

(i) $y(x) = 3^{x^2 - x}$

$3^{x^2-x} = e^{\ln 3^{x^2-x}} = e^{(x^2-x)\cdot\ln 3} = e^{\ln 3\cdot(x^2-x)}$

$$\frac{d(e^{\ln 3\cdot(x^2-x)})}{dx} = \frac{d(e^{\ln 3(x^2-x)})}{d(\ln 3(x^2-x))} \cdot \frac{d(\ln 3(x^2-x))}{d(x^2-x)} \cdot \frac{d(x^2-x)}{dx}$$

$$= e^{\ln 3(x^2-x)} \cdot \ln 3 \cdot (2x - 1)$$

$$= 3^{x^2-x} \cdot \ln 3 \cdot (2x - 1)$$

In general, if $y = a^{f(x)}$, then $y' = a^{f(x)} \cdot \ln a \cdot f'(x)$.

2. Verify the derivatives from the chart:

(a) $y = \csc x = \dfrac{1}{\sin x} = (\sin x)^{-1}$

$$y' = -1(\sin x)^{-2}(\cos x)$$
$$= -1 \cdot \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$$
$$= -\csc x \cot x$$

(b) $y = \sec x = \dfrac{1}{\cos x} = (\cos x)^{-1}$

$$y' = -1(\cos x)^{-2}(-\sin x)$$
$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$
$$= \sec x \tan x$$

(c) $y = \cot x = \dfrac{\cos x}{\sin x}$

$$y' = \frac{-\sin x \sin x - \cos x \cos x}{(\sin x)^2}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\csc^2 x$$

(d) $y = \log_a x$

$$\begin{aligned}
a^y &= x \\
a^{\log_a x} &= x \\
\frac{d\left(a^{\log_a x}\right)}{dx} &= \frac{dx}{dx} \\
\frac{d\left(a^{\log_a x}\right)}{d\left(\log_a x\right)} \cdot \frac{d\left(\log_a x\right)}{dx} &= \frac{dx}{dx} \\
a^{\log_a x} \ln a \cdot \frac{d\left(\log_a x\right)}{dx} &= 1 \\
\frac{d\left(\log_a x\right)}{dx} &= \frac{1}{a^{\log_a x} \ln a} = \frac{1}{x \ln a}
\end{aligned}$$

But $\dfrac{d\left(\log_a x\right)}{dx}$ is just $\dfrac{dy}{dx}$ with different notation!

$$\therefore y' = \frac{1}{x \ln a}$$

3. Find the equation of the tangent at the given point.

(a) $f(x) = e^x \cos x \qquad P(0, 1)$

$$\begin{aligned}
f'(x) &= e^x \cos x + e^x(-\sin x) \\
&= e^x(\cos x - \sin x)
\end{aligned}$$

$m = f'(0) = e^0(\cos 0 - \sin 0) = 1(1 - 0) = 1$

$y = 1, \quad x = 0, \quad m = 1, \quad b = ?$

$1 = 1(0) + b \implies b = 1$

The equation of the tangent at the given point is $y = x + 1$.

(b) $g(x) = \cos x - \sin x \qquad P(\pi, -1)$

$g'(x) = -\sin x - \cos x$

$m = g'(\pi) = -\sin \pi - \cos \pi = 0 - (-1) = 1$

$y = -1, \quad x = \pi, \quad m = 1, \quad b = ?$

$-1 = 1(\pi) + b \Longrightarrow b = -(\pi + 1)$

The equation of the tangent at the given point is $y = x - (\pi + 1)$.

(c) $h(x) = 2^x \qquad P(0, 1)$

$h'(x) = 2^x \ln 2$

$m = h'(0) = 2^0 \ln 2 = \ln 2$

$y = 1, \quad x = 0, \quad m = \ln 2, \quad b = ?$

$1 = \ln 2(0) + b \Longrightarrow b = 1$

The equation of the tangent at the given point is $y = (\ln 2)x + 1$.

(d) $G(x) = xe^{-x^2} \qquad P(0, 0)$

$$G'(x) = (1)e^{-x^2} + xe^{-x^2}(-2x)$$
$$= e^{-x^2}(1 - 2x^2)$$

$m = G'(0) = e^0(1 - 0) = 1$

$y = 0, \quad x = 0, \quad m = 1, \quad b = ?$

$0 = 1(0) + b \Longrightarrow b = 0$

The equation of the tangent at the given point is $y = x$.

4. Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

$$r'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$
$$r'(1) = f'(3) \cdot g'(2) \cdot 4$$
$$= 6 \cdot 5 \cdot 4$$
$$= 120$$

5. For what values of r does $y = e^{rx}$ satisfy the differential equation $y'' + y' - 6y = 0$?

$$y' = re^{rx}$$
$$y'' = r^2 e^{rx}$$

$$y'' + y' - 6y = 0$$
$$r^2 e^{rx} + re^{rx} - 6e^{rx} = 0$$
$$e^{rx}(r^2 + r - 6) = 0$$
$$e^{rx}(r + 3)(r - 2) = 0$$

e^{rx} is never zero. Therefore we have $r = -3$ or $r = 2$.

6. Find the 50th derivative of $y = \cos 2x$.

$$\begin{aligned} f(x) &= \cos 2x \\ f^{(1)}(x) &= -2\sin 2x \\ f^{(2)}(x) &= -2^2\cos 2x \\ f^{(3)}(x) &= 2^3\sin 2x \\ f^{(4)}(x) &= 2^4\cos 2x \end{aligned}$$

From this pattern, we can define the nth derivative of $f(x)$ as:

$$f^{(n)}(x) = \begin{cases} 2^n\cos 2x & n \bmod 4 = 0 \\ -2^n\sin 2x & n \bmod 4 = 1 \\ -2^n\cos 2x & n \bmod 4 = 2 \\ 2^n\sin 2x & n \bmod 4 = 3 \end{cases}$$

where $n \bmod 4$ is the remainder from dividing n by 4.

Since $50 \bmod 4 = 2$, then $f^{(50)}(x) = -2^{50}\cos 2x$.