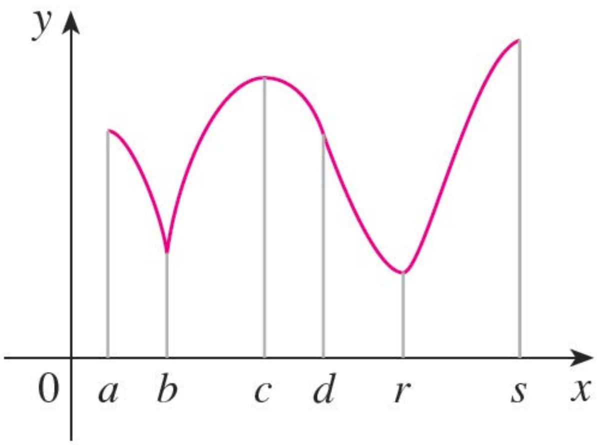
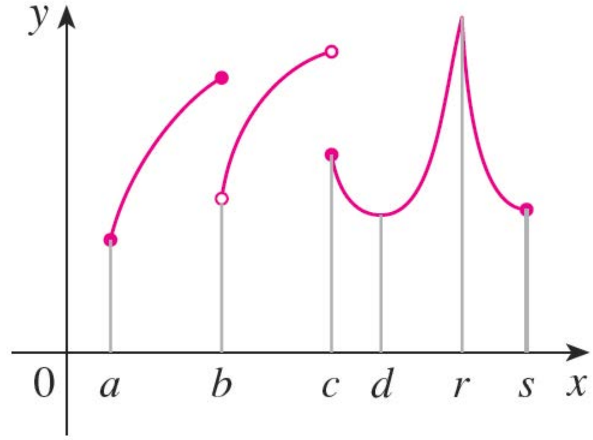
Tutorial 8

Week of November 5, 2018

1. For each of the following numbers, a, b, c, d, r, and s, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.



(a)



(b)

(a) a: nothing

b: local minimum

c: local maximum

d: nothing

r: local minimum and absolute minimum

s: absolute maximum

(b) a: absolute minimum

b: local maximum

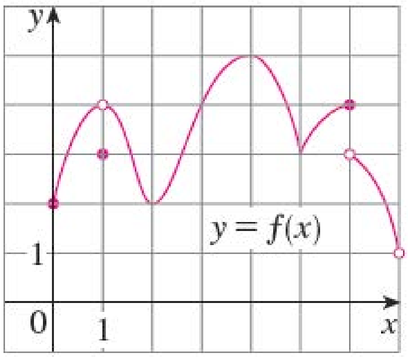
c: nothing

d: local minimum

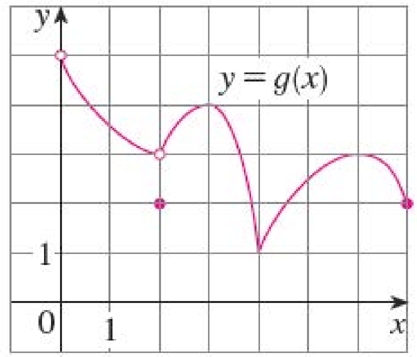
r: absolute maximum

s: nothing

2. For each of the following graphs, state the points where the local and global extrema occur.



(a)



(b)

(a)
- Local minimum at $(1, 3)$
- Local minimum at $(2, 2)$
- Local and absolute maximum at $(4, 5)$
- Local minimum at $(5, 3)$
- Local maximum at $(6, 4)$
- There is no absolute minimum

(b)
- Local minimum at $(2, 2)$
- Local maximum at $(3, 4)$
- Local minimum and absolute minimum at $(4, 1)$
- Local maximum at $(6, 3)$
- There is not absolute maximum

3. Find the critical numbers of the given functions.

(a) $f(x) = x^3 + 6x^2 - 15x$

$$\begin{aligned} f'(x) &= 3x^2 + 12x - 15 \\ &= 3(x^2 + 4x - 5) \\ &= 3(x + 5)(x - 1) \end{aligned}$$

Setting $f'(x) = 0$ we get:

$x_1 = -5, \quad x_2 = 1$

(b) $f(x) = 2x^3 - 3x^2 - 36x$

$$\begin{aligned} f'(x) &= 6x^2 - 6x - 36 \\ &= 6(x^2 - x - 6) \\ &= 6(x-3)(x+2) \end{aligned}$$

Setting $f'(x) = 0$ we get:

$x_1 = 3, \quad x_2 = -2$

(c) $g(t) = |3t - 4|$

$$g(t) = \begin{cases} 3t-4 & 3t-4 \geq 0 \\ -(3t-4) & 3t-4 < 0 \end{cases}$$

$$= \begin{cases} 3t-4 & t \geq \frac{4}{3} \\ -(3t-4) & t < \frac{4}{3} \end{cases}$$

$$g'(t) = \begin{cases} 3 & t > \frac{4}{3} \\ -3 & t < \frac{4}{3} \end{cases}$$

$g'(t)$ is never zero. $g'(\frac{4}{3})$ does not exist so $\frac{4}{3}$ is the only critical point. (This is where the "corner" of the function is located.)

(d) $h(p) = \dfrac{p-1}{p^2+4}$

$$h'(p) = \frac{(1)(p^2+4) - (p-1)(2p)}{(p^2+4)^2} = \frac{p^2+4-2p^2+2p}{(p^2+4)^2} = \frac{-(p^2-2p-4)}{(p^2+4)^2}$$

$h'(p) = 0$ when $p^2 - 2p - 4 = 0$. Using quadratic formula, we obtain $p_1 = 1 + \sqrt{5}$ and $p_2 = 1 - \sqrt{5}$.

The denominator of $h'(p)$ is never zero so we do not need to worry about $h'(p)$ not existing. We have a total of two critical numbers.

(e) $g(x) = \sqrt[3]{4-x^2} = (4-x^2)^{\frac{1}{3}}$

$$g'(x) = \frac{1}{3}(4-x^2)^{-\frac{2}{3}}(-2x) = \frac{-2x}{3\sqrt[3]{(4-x^2)^2}}$$

$g'(x) = 0$ when $x = 0$.

$g'(x)$ does not exist when $\sqrt[3]{(4-x^2)^2} = 0$

$\implies 4 - x^2 = 0$

$x_1 = 2, \quad x_2 = -2$

There is a total of 3 critical numbers: $-2, 0, 2$.

4. Find the absolute maximum and minimum of the following functions on the given interval.

(a) $f(x) = 3x^4 - 4x^3 - 12x^2 + 1, \quad [-2, 3]$

$$\begin{aligned} f'(x) &= 12x^3 - 12x^2 - 24x \\ &= 12x(x^2 - x - 2) \\ &= 12x(x-2)(x+1) \end{aligned}$$

The critical numbers are $-1, 0, 2$, all of which are inside the given interval.

- $f(-2) = 33$
- $f(-1) = -4$
- $f(0) = 1$
- $f(2) = -31$
- $f(3) = 28$

We have an absolute maximum at the point $(-2, 33)$ and an absolute minimum at the point $(2, -31)$.

(b) $f(t) = (t^2 - 4)^3, \quad [-2, 3]$

$f'(t) = 3(t^2 - 4)^2(2t)$

The critical numbers are $-2, 0, 2$, all of which are inside the given interval.

- $f(-2) = 0$
- $f(0) = -64$
- $f(2) = 0$
- $f(3) = 125$

We have an absolute maximum at the point $(3, 125)$ and an absolute minimum at the point $(0, -64)$.

(c) $f(x) = \dfrac{x}{x^2 - x + 1}, \quad [0, 3]$

$$\begin{aligned} f'(x) &= \frac{(1)(x^2 - x + 1) - (x)(2x - 1)}{(x^2 - x + 1)^2} = \frac{x^2 - x + 1 - 2x^2 + x}{(x^2 - x + 1)^2} \\ &= \frac{-x^2 + 1}{(x^2 - x + 1)^2} = \frac{-(x^2 - 1)}{(x^2 - x + 1)^2} \end{aligned}$$

$f'(x) = 0$ when $x_1 = 1$ or $x_2 = -1$. The denominator has no real roots so $f'(x)$ will always exist. We have a total of two critical numbers.

- $f(-1) = -\frac{1}{3}$
- $f(0) = 0$
- $f(1) = 1$
- $f(3) = \frac{3}{7}$

We have an absolute maximum at the point $(1, 1)$ and an absolute minimum at the point $(-1, -\frac{1}{3})$.

5. For each of the following functions:

   (i) Find the intervals of increase and decrease

   (ii) Find the values of the local maximum and minimum

   (iii) Find the intervals of concavity and inflection points

   (a) $f(x) = x^3 - 3x^2 - 9x + 4$

   $$\begin{aligned} f'(x) &= 3x^2 - 6x - 9 \\ &= 3(x^2 - 2x - 3) \\ &= 3(x-3)(x+1) \end{aligned}$$

   The critical numbers are -1 and 3.

   i.

	$x < -1$	$-1 < x < 3$	$x > 3$
$3(x-3)(x+1)$	$+$	$-$	$+$

   - f is increasing on the interval $(-\infty, -1)$
   - f is decreasing on the interval $(-1, 3)$
   - f is increasing on the interval $(3, \infty)$

   ii. $f''(x) = 6x - 6 = 6(x-1)$

   $f''(-1) = 6(-1-1) = -12$. f is concave down at $x = -1$. $f(-1) = 9$ is a local maximum.

   $f''(3) = 6(3-1) = 12$. f is concave down at $x = 3$. $f(3) = -23$ is a local minimum.

   iii. $f''(x) = 6(x-1)$

   $f''(x) > 0$ when $6(x-1) > 0 \Longrightarrow x > 1$. $f(x)$ is concave up on the interval $(1, \infty)$.

   $f''(x) < 0$ when $6(x-1) < 0 \Longrightarrow x < 1$. $f(x)$ is concave down on the interval $(-\infty, 1)$.

   $f''(x) = 0$ when $x = 1$. Since $x = 1$ was not a critical number of f, we have an inflection point at $(1, -7)$. Alternatively, we see that f changes concavity on the left and right of $x = 1$.

   (b) $f(x) = 2x^3 - 9x^2 + 12x - 3$

   $$\begin{aligned} f'(x) &= 6x^2 - 18x + 12 \\ &= 6(x^2 - 3x + 2) \\ &= 6(x-2)(x-1) \end{aligned}$$

   The critical numbers are 1 and 2.

   i.

	$x < 1$	$1 < x < 2$	$x > 2$
$6(x-2)(x-1)$	$+$	$-$	$+$

   - f is increasing on the interval $(-\infty, 1)$

- f is decreasing on the interval $(1, 2)$
- f is increasing on the interval $(2, \infty)$

ii. $f''(x) = 12x - 18$

$f''(1) = 12 - 18 = -6$. f is concave down at $x = 1$. $f(1) = 2$ is a local maximum.

$f''(2) = 24 - 18 = 6$. f is concave down at $x = 2$. $f(2) = 1$ is a local minimum.

iii. $f''(x) = 12x - 18$

$f''(x) > 0$ when $12x - 18 > 0 \Longrightarrow x > \frac{3}{2}$. $f(x)$ is concave up on the interval $(\frac{3}{2}, \infty)$.

$f''(x) < 0$ when $12x - 18 < 0 \Longrightarrow x < \frac{3}{2}$. $f(x)$ is concave down on the interval $(-\infty, \frac{3}{2})$.

$f''(x) = 0$ when $x = \frac{3}{2}$. Since $x = \frac{3}{2}$ was not a critical number of f, we have an inflection point at $(\frac{3}{2}, \frac{3}{2})$. Alternatively, we can see that f changes concavity on the left and right of $x = \frac{3}{2}$.

(c) $f(x) = x^2 \ln x$ Domain: $x > 0$!!

$$\begin{aligned} f'(x) &= 2x \ln x + \frac{x^2}{x} \quad \text{(Since } x \neq 0 \text{ so we can cancel it)} \\ &= 2x \ln x + x \\ &= x(2 \ln x + 1) \end{aligned}$$

i.
$$\begin{aligned} f'(x) &< 0 \\ x(2\ln x + 1) &< 0 \quad \text{(Since } x \neq 0 \text{ we can divide both sides by it)} \\ 2 \ln x + 1 &< 0 \\ \ln x &< -\frac{1}{2} \\ x &< e^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} f'(x) &> 0 \\ x(2\ln x + 1) &> 0 \quad \text{(Since } x \neq 0 \text{ we can divide both sides by it)} \\ 2 \ln x + 1 &> 0 \\ \ln x &> -\frac{1}{2} \\ x &> e^{-\frac{1}{2}} \end{aligned}$$

- f is decreasing on the interval $(0, e^{-\frac{1}{2}})$
- f is increasing on the interval $(e^{-\frac{1}{2}}, \infty)$

Clearly, $f'(x) = 0$ when $x = e^{-\frac{1}{2}}$. So we have one critical number, $e^{-\frac{1}{2}}$.

Something worth nothing is that the act of raising both sides as a power of e preserves order. This is because e^x is a monotonically increasing function. If $x < y$ then $e^x < e^y$. If $x > y$ then $e^x > e^y$.

ii. We have found in the first part that f is decreasing left of $e^{-\frac{1}{2}}$ and increasing right of $e^{-\frac{1}{2}}$. Therefore, $f(e^{-\frac{1}{2}}) = -\frac{1}{2}e^{-1}$ is a local minimum. There is no local maximum.

iii. $$\begin{aligned} f''(x) &= 2\left(\ln x + x \cdot \frac{1}{x}\right) + 1 \\ &= 2\ln x + 2 + 1 \\ &= 2\ln x + 3 \end{aligned}$$

$f''(x) > 0$ when $2\ln x + 3 > 0 \Longrightarrow x > e^{-\frac{3}{2}}$. $f(x)$ is concave up on the interval $(e^{-\frac{3}{2}}, \infty)$.

$f''(x) < 0$ when $2\ln x + 3 < 0 \Longrightarrow x < e^{-\frac{3}{2}}$. $f(x)$ is concave down on the interval $(0, e^{-\frac{3}{2}})$.

$f''(x) = 0$ when $x = e^{-\frac{3}{2}}$. Since this is not a critical number of f, we have an inflection point at $(e^{-\frac{3}{2}}, -\frac{3}{2}e^{-3})$. Alternatively, we notice that the concavity changes from the left to the right of $e^{-\frac{3}{2}}$.

6. For the following function, find:
   (a) The vertical and horizontal asymptotes
   (b) The intervals of increase and decrease
   (c) The values of the local maximum and minimum
   (d) The intervals of concavity and inflection points

$$f(x) = e^{-x^2}$$

(a) Vertical asymptotes: $e^{-x^2} = \frac{1}{e^{x^2}}$. Since the exponential function is never zero for finite x, $\frac{1}{e^{x^2}}$ will never blow up to infinity. We conclude that there are no vertical asymptotes.

Horizontal asymptotes:

$$\lim_{x\to\infty} e^{-x^2} = 0$$

$$\lim_{x\to-\infty} e^{-x^2} = 0$$

We have a horizontal asymptote at $y = 0$ as x goes off to positive and negative infinity.

(b) $f'(x) = e^{-x^2}(-2x) = -2xe^{-x^2}$

$f'(x) < 0$ when $-2xe^{-x^2} < 0$. e^{-x^2} is always positive (and non-zero). Therefore we have that $-2x < 0 \Longrightarrow x > 0$ (Must flip signs when dividing both sides by -2).

Similarly, we get that $f'(x) > 0$ when $x < 0$. Therefore our function is increasing on the interval $(-\infty, 0)$ and decreasing on the interval $(0, \infty)$.

(c) Set $f'(x) = 0$. e^{-x^2} is never zero, which implies that $-2x$ must be zero. This occurs when $x = 0$. So we have one critical number. In (b), we found that our function was increasing on the left of $x = 0$ and decreasing on the right of $x = 0$. Therefore $f(0) = 1$ is a local maximum. There is no local minimum.

(d) $f''(x) = -2\left((1)e^{-x^2} + xe^{-x^2}(-2x)\right) = -2e^{-x^2}(1-2x^2)$

$f''(x) = 0$ when $-2(1-2x^2) = 0$. Solving, we get that $x_1 = \dfrac{1}{\sqrt{2}}$ and $x_2 = -\dfrac{1}{\sqrt{2}}$.

	$x < -\dfrac{1}{\sqrt{2}}$	$-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$	$x > \dfrac{1}{\sqrt{2}}$
$-2e^{-x^2}(1-2x^2)$	$+$	$-$	$+$

The function is concave up on the intervals $\left(-\infty, -\dfrac{1}{\sqrt{2}}\right)$ and $\left(\dfrac{1}{\sqrt{2}}, \infty\right)$. The function is concave down on the interval $\left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$. Since neither x_1 nor x_2 were critical numbers of f , we conclude that we have inflection points when $x_1 = \dfrac{1}{\sqrt{2}}$ and $x_2 = -\dfrac{1}{\sqrt{2}}$. Alternatively, we could have also just used the computed intervals of concavity.