Tutorial 9

Week of November 12, 2018

1. Sketch the following curves. Consider domain, intercepts, asymptotes, intervals of increase/ decrease, local extrema, intervals of concavity, and inflection points.

   (a) $f(x) = x^3 + 3x^2$

   Domain: $\mathcal{D} = \{x \in \mathbb{R}\}$

   Y-Intercept: Set $x = 0$. Then $y = 0$. The y-intercept is $(0, 0)$.

   X-Intercept: Set $y = 0$. Then $x = 0$ and $x = 3$ The x-intercepts are $(0, 0)$ and $(3, 0)$.

   Vertical Asymptotes: Looking for places where function is undefined. There are none.

   Horizontal Asymptotes: Evaluate the limit of $f(x)$ as x approaches $\pm\infty$. There are no horizontal asymptotes.

   Intervals of Increase/Decrease: $f'(x) = 3x^2 + 6x = 3x(x + 2)$. $f'(x) = 0$ when $x = 0$ and $x = -2$.

	$x < -2$	$-2 < x < 0$	$x > 0$
$3x$	$-$	$-$	$+$
$x + 2$	$-$	$+$	$+$
$f'(x)$	$+$	$-$	$+$

   f is increasing on $(-\infty, -2)$ and $(0, \infty)$. f is decreasing on $(-2, 0)$.

   Since f is decreasing left of 0 and increasing right of 0, a local minimum occurs at $x = 0$. Since f is increasing left of -2 and decreasing right of -2, a local maximum occurs at $x = -2$.

   Intervals of Concavity and Inflection Points:

   $f''(x) = 6x + 6 = 6(x + 1)$. $f''(x) = 0$ when $x = -1$. $f''(x)$ is less than zero when $x < -1$ (concave down). $f''(x)$ is greater than zero when $x > -1$ (concave up). We have an inflection point at $x = 1$. Without calculating intervals of concavity, since $x = 1$ was not a critical number, we could have concluded from here that there would be an inflection point at $x = 1$.

   (b) $f(x) = \dfrac{x}{x - 1}$

   Domain: $\mathcal{D} = \{x \in \mathbb{R} \mid x \neq 1\}$

   Y-Intercept: Set $x = 0$. Then $y = 0$. The y-intercept is $(0, 0)$.

   X-Intercept: Set $y = 0$. Then $x = 0$. The x-intercept is also $(0, 0)$.

   Vertical Asymptotes: $\lim_{x \to 1^-} \dfrac{x}{x - 1} = -\infty$

   $\lim_{x \to 1^+} \dfrac{x}{x - 1} = +\infty$

   We have a vertical asymptote at $x = 1$.

Horizontal Asymptotes:

$$\lim_{x\to-\infty} \frac{x}{x-1} = \lim_{x\to-\infty} \frac{x\left(\frac{1}{x}\right)}{(x-1)\left(\frac{1}{x}\right)} = \lim_{x\to-\infty} \frac{1}{1-\frac{1}{x}} = 1$$

Similarly, $\lim\limits_{x\to\infty} \dfrac{x}{x-1} = 1$.

We have a horizontal asymptote at $y = 1$.

Intervals of Increase/Decrease:

$$f'(x) = \frac{(1)(x-1)-(x)(1)}{(x-1)^2} = \frac{x-1-x}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

We notice that the denominator is never zero since -1 is not in the domain. In addition, since it is a square, it is always positive. Since the numerator is negative, $f'(x)$ will always be less than zero. This means that $f(x)$ is always decreasing.

$f''(x) = 2(x-1)^{-3}$. $f''(x)$ is never zero so we don't have any inflection points.

$f''(x) < 0$ when $x - 1 < 0 \Longrightarrow x < 1$. So $f(x)$ is concave down when $x < 1$. Similarly, $f''(x) > 0$ when $x - 1 > 0 \Longrightarrow x > 1$. So $f(x)$ is concave up when $x > 1$.

2. Find two positive numbers whose product is 100 and whose sum is a minimum.

   Let x and y be the two numbers. Then we get $xy = 100 \Longrightarrow y = \frac{100}{x}$. We want to minimize the sum $f(x) = x + y = x + \frac{100}{x}$.

   $f'(x) = 1 - \frac{100}{x^2}$

   Set $f'(x) = 0$. Solving, we get that $x = 10$. (Note that we were told the numbers were positive and they definitely can't be zero since we have a product of 100).

   $f''(x) = \frac{200}{x^3}$. $f''(10) = \frac{1}{5} > 0$ (concave up) which confirms that a minimum occurs at $x = 10$. Therefore our two numbers are $x = 10$ and $y = 10$.

3. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

   Let the point on the curve be (x, y). The distance from the point (x, y) to the point $(3, 0)$ can be represented by:

   $$\begin{aligned} d(x) &= \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(3-x)^2 + (0-y)^2} \\ &= \sqrt{(3-x)^2 + (0-\sqrt{x})^2} = \sqrt{9 - 6x + x^2 + x} \\ &= \sqrt{x^2 - 5x + 9} \end{aligned}$$

   Since we want to the find the point on the curve that is closest to $(3, 0)$, we want to minimize $d(x)$.

   $$d'(x) = \frac{1}{2}(x^2 - 5x + 9)^{-\frac{1}{2}}(2x - 5) = \frac{2x-5}{2\sqrt{x^2 - 5x + 9}}$$

   Set $d'(x) = 0$. Then $2x - 5 = 0 \Longrightarrow x = \frac{5}{2}$.

   $$d''(x) = \frac{1}{2}\left(\frac{(2)(x^2-5x+9)^{\frac{1}{2}} - (2x-5)\frac{1}{2}(x^2-5x+9)^{-\frac{1}{2}}(2x-5)}{(\sqrt{x^2-5x+9})^2}\right)$$

Notice that when we evaluate $d''(\frac{5}{2})$, the second term on the numerator will drop out, the factor of 2 in front of the first term of the numerator will cancel with the 1/2 out front, and $(x^2-5x+9)^{\frac{1}{2}}$ will cancel with one of itself in the denominator leaving you with:

$$d''\left(\frac{5}{2}\right) = \frac{1}{\sqrt{\frac{25}{4}-\frac{25}{2}+9}} = \frac{1}{\sqrt{\frac{25}{4}-\frac{50}{4}+\frac{36}{4}}} = \frac{1}{\sqrt{\frac{11}{4}}} > 0$$

The function is concave up at $\frac{5}{2}$ meaning we have found a minimum. Plugging $x=\frac{5}{2}$ into $y=\sqrt{x}$, we get that our point should be $\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$ in order to minimize the distance to the point $(3,0)$.

4. For the following vectors, find $\mathbf{a}+\mathbf{b}$, $4\mathbf{a}+2\mathbf{b}$, $|\mathbf{a}|$, and $|\mathbf{a}-\mathbf{b}|$.

   (a) $\mathbf{a} = \langle -3, 4\rangle$, $\mathbf{b} = \langle 9, -1\rangle$

   $\mathbf{a}+\mathbf{b} = \langle -3+9, 4-1\rangle = \langle 6, 3\rangle$

   $4\mathbf{a} = \langle -12, 16\rangle$, $2\mathbf{b} = \langle 18, -2\rangle$. $4\mathbf{a}+2\mathbf{b} = \langle -12+18, 16-2\rangle = \langle 6, 14\rangle$.

   $|\mathbf{a}| = \sqrt{(-3)^2+(4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

   $|\mathbf{a}-\mathbf{b}| = \sqrt{(-3-9)^2+(4+1)^2} = \sqrt{144+25} = \sqrt{169} = 13.$

   (b) $\mathbf{a} = 4\mathbf{i}-3\mathbf{j}+2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i}-4\mathbf{k}$

   $\mathbf{a}+\mathbf{b} = (4+2)\mathbf{i}+(-3+0)\mathbf{j}+(2-4)\mathbf{k} = 6\mathbf{i}-3\mathbf{j}-2\mathbf{k}$

   $4\mathbf{a} = 16\mathbf{i}-12\mathbf{j}+4\mathbf{k}$, $2\mathbf{b} = 4\mathbf{i}-8\mathbf{k}$

   $4\mathbf{a}+2\mathbf{b} = (16+4)\mathbf{i}+(-12+0)\mathbf{j}+(4-8)\mathbf{k} = 20\mathbf{i}-12\mathbf{j}-4\mathbf{k}$

   $|\mathbf{a}| = \sqrt{(4)^2+(-3)^2+(2)^2} = \sqrt{16+9+4} = \sqrt{29}$

   $|\mathbf{a}-\mathbf{b}| = \sqrt{(4-2)^2+(-3-0)^2+(2+4)^2} = \sqrt{4+9+36} = \sqrt{49} = 7$

5. Find a unit vector that has the same direction as the given vector.

   (a) $\langle 6, -2\rangle$

   This vector has length $\sqrt{(6)^2+(-2)^2} = \sqrt{40} = 2\sqrt{10}$. Then a unit vector in the same direction would be $\frac{1}{2\sqrt{10}}\langle 6, -2\rangle = \langle \frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\rangle$.

   (b) $-5\mathbf{i}+3\mathbf{j}-\mathbf{k}$

   This vector has length $\sqrt{(-5)^2+(3)^2+(-1)^2} = \sqrt{25+9+1} = \sqrt{35}$. Then a unit vector in the same direction would be $-\frac{5}{\sqrt{35}}\mathbf{i}+\frac{3}{\sqrt{35}}\mathbf{j}-\frac{1}{\sqrt{35}}\mathbf{k}$.

6. Find a vector in the same direction as $\mathbf{v} = \langle 6, 2, -3\rangle$ with length 4.

   This vector has length $\sqrt{(6)^2+(2)^2+(-3)^2} = \sqrt{36+4+9} = \sqrt{49} = 7$. We can find a vector in the same direction with length 4 by multiplying the vector by $\frac{4}{7}$. The new vector will be $\frac{4}{7}\langle 6, 2, -3\rangle = \langle \frac{24}{7}, \frac{8}{7}, \frac{-12}{7}\rangle$.