Tutorial 3 Solutions

Question 1

(10.2) An experimenter has prepared a drug dosage level that she claims will induce sleep for 80% of people suffering from insomnia. After examining the dosage, we feel that her claims regarding the effectiveness of the dosage are inflated. In an attempt to disprove her claim, we administer her prescribed dosage to 20 insomniacs and we observe Y, the number for whom the drug dose induces sleep. We wish to test the hypotheses:

$$H_0 : p = 0.8 \quad \text{vs} \quad H_1 : p < 0.8.$$

Assume that the rejection region $\{y \leq 12\}$ is used.

(a) In the context of this problem, what is a type I error?

A type I error occurs when we incorrectly reject the null hypothesis in favour of the alternative hypothesis, despite the null hypothesis actually being true. In the context of this problem, a type I error would be to conclude that less than 80% of insomniacs respond to the drug, when the drug actually induces sleep in 80% of insomniacs.

(b) Find α, the probability of committing a type I error.

The probability of committing a type I error is the probability of being inside the rejection region, under the condition that the null hypothesis is true. As such, we have:

$$\begin{aligned} \alpha &= \mathbf{P}\left(\text{Reject } H_0 \,|\, H_0 \text{ true}\right) \\ &= \mathbf{P}\left(Y \leq 12 \,|\, p = 0.8\right) \\ &= 0.0321 \end{aligned}$$

Note that the above probability can be found in **R** using the following code:

```r
pbinom(12, size=20, prob=0.8)
```

```
## [1] 0.03214266
```

(c) In the context of this problem, what is a type II error?

A type II error occurs when we fail to reject the null hypothesis in favour of the alternative hypothesis, despite the null hypothesis actually being false. In the context of this problem, a type II error would be to conclude that the drug induces sleep in 80% of insomniacs, when the drug actually induces sleep in less than 80% of insomniacs.

(d) Find β, the probability of committing a type II error, when $p = 0.6$.

The probability of committing a type II error requires that we are outside the rejection region despite $p = 0.6 < 0.8$. As such, we have:

$$\beta(0.6) = \mathbf{P}\left(\text{fail to reject } H_0 \,|\, H_0 \text{ false}\right)$$

$$
\begin{aligned}
&= \mathbf{P}\left(Y > 12 \,|\, p = 0.6\right) \\
&= 1 - \mathbf{P}\left(Y \leq 12 \,|\, p = 0.6\right) \\
&= 0.4159
\end{aligned}
$$

Note that the above probability can be found in **R** using the following code:

```
1 - pbinom(12, size=20, prob=0.6)
```

```
## [1] 0.4158929
```

```
## OR ##
pbinom(12, size=20, prob=0.6, lower.tail=FALSE)
```

```
## [1] 0.4158929
```

(e) Find β, the probability of committing a type II error, when $p = 0.4$.

The probability of committing a type II error requires that we are outside the rejection region despite $p = 0.4 < 0.8$. As such, we have:

$$
\begin{aligned}
\beta(0.4) &= \mathbf{P}\left(\text{fail to reject } H_0 \,|\, H_0 \text{ false}\right) \\
&= \mathbf{P}\left(Y > 12 \,|\, p = 0.4\right) \\
&= 1 - \mathbf{P}\left(Y \leq 12 \,|\, p = 0.4\right) \\
&= 0.0210
\end{aligned}
$$

Note that the above probability can be found in **R** using the following code:

```
1 - pbinom(12, size=20, prob=0.4)
```

```
## [1] 0.02102893
```

```
## OR ##
pbinom(12, size=20, prob=0.4, lower.tail=FALSE)
```

```
## [1] 0.02102893
```

Question 2

(10.94) Suppose that $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a normal distribution with known mean μ and unknown variance σ^2. Find the most powerful α-level test of

$$
H_0 : \sigma^2 = \sigma_0^2 \quad \text{vs} \quad H_1 : \sigma^2 = \sigma_1^2,
$$

where $\sigma_1^2 > \sigma_0^2$. Show that this test is equivalent to a χ^2 test.

The form of the rejection region for the most powerful α-level test is found using the Neyman-Pearson Lemma (Theorem 10.1, page 542 in the textbook). As we have a random sample of size n, the likelihood functions are obtained as the product of the marginal densities under each of H_0 and H_1.

$$
\frac{\mathcal{L}(\sigma_0)}{\mathcal{L}(\sigma_1)} = \frac{(2\pi)^{-n/2}\,\sigma_0^{-n}\,\exp\left\{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(Y_i - \mu)^2\right\}}{(2\pi)^{-n/2}\,\sigma_1^{-n}\,\exp\left\{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n}(Y_i - \mu)^2\right\}}
$$

$$
\begin{aligned}
&= \left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\left\{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(Y_i-\mu)^2 + \frac{1}{2\sigma_1^2}\sum_{i=1}^{n}(Y_i-\mu)^2\right\} \\
&= \left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\left\{\left(\frac{1}{2\sigma_1^2}-\frac{1}{2\sigma_0^2}\right)\sum_{i=1}^{n}(Y_i-\mu)^2\right\} \\
&= \left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\left\{\frac{\sigma_0^2-\sigma_1^2}{2\sigma_1^2\sigma_0^2}\sum_{i=1}^{n}(Y_i-\mu)^2\right\} \\
&< k,
\end{aligned}
$$

where k is an arbitrary constant. Starting from the last two lines above, we can take the logarithm of both sides in order to reach an inequality concerning a statistic $T(Y_1, Y_2, \ldots, Y_n)$, which is a function of our data.

$$
\begin{aligned}
\left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\left\{\frac{\sigma_0^2-\sigma_1^2}{2\sigma_1^2\sigma_0^2}\sum_{i=1}^{n}(Y_i-\mu)^2\right\} &< k \\
n\log\left(\frac{\sigma_1}{\sigma_0}\right) + \frac{\sigma_0^2-\sigma_1^2}{2\sigma_1^2\sigma_0^2}\sum_{i=1}^{n}(Y_i-\mu)^2 &< \log(k) \\
n\log\left(\frac{\sigma_1}{\sigma_0}\right) - \log(k) &< -\left(\frac{\sigma_0^2-\sigma_1^2}{2\sigma_1^2\sigma_0^2}\sum_{i=1}^{n}(Y_i-\mu)^2\right) \\
n\log\left(\frac{\sigma_1}{\sigma_0}\right) - \log(k) &< \frac{\sigma_1^2-\sigma_0^2}{2\sigma_1^2\sigma_0^2}\sum_{i=1}^{n}(Y_i-\mu)^2 \\
\frac{2\sigma_1^2\sigma_0^2}{\sigma_1^2-\sigma_0^2}\left(n\log\left(\frac{\sigma_1}{\sigma_0}\right) - \log(k)\right) &< \sum_{i=1}^{n}(Y_i-\mu)^2
\end{aligned}
$$

Equivalently, this means that

$$
T = \sum_{i=1}^{n}(Y_i-\mu)^2 > \frac{2\sigma_1^2\sigma_0^2}{\sigma_1^2-\sigma_0^2}\left(n\log\left(\frac{\sigma_1}{\sigma_0}\right) - \log(k)\right),
$$

which is to say that we should reject the null hypothesis if the statistic T is large. To find a rejection region of size α, we note that

$$
\frac{T}{\sigma_0^2} = \frac{\sum_{i=1}^{n}(Y_i-\mu)^2}{\sigma_0^2} = \sum_{i=1}^{n}\left(\frac{Y_i-\mu}{\sigma_0}\right)^2,
$$

has a chi-square distribution on n degrees of freedom. It follows that the most powerful α-level test is equivalent to the chi-square test.

Question 3

Generate $n = 30$ observations, $X_1, X_2, \ldots, X_n$, from a Bernoulli distribution with parameter $p = 0.4$. Consider the problem of testing

$$H_0 : p = 0.3 \quad \text{vs.} \quad H_1 : p > 0.3,$$

by means of a test that rejects H_0 for *large* values of the test statistic

$$T(X_1, X_2, \dots, X_n) = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{\hat{p} - 0.3}{\sqrt{0.3(1-0.3)/n}},$$

where $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample proportion.

(a) Do you reject H_0 at level $\alpha = 0.05$?

As usual, for simulations, we should set a seed.

```
set.seed(20)
```

Note that to generate 30 observations from the Bernoulli distribution is equivalent to generating one observation from a Binomial distribution with a size value of 30. As the result of `rbinom()` is the number of successes out of 30, we scale it immediately in order to obtain the proportion of successes.

```
phat <- rbinom(n=1, size=30, prob=0.4) / 30

phat
```

```
## [1] 0.5
```

The value of the test statistic is:

```
p0 <- 0.3

statistic <- (phat - p0) / sqrt(p0 * (1 - p0) / 30)

statistic
```

```
## [1] 2.390457
```

As this is an upper-tailed test, we should reject the null hypothesis for large values of the test statistic. At the 5% significance level, our rejection region is of the form

$$\{z > z_{\alpha=0.05} = 1.645\}.$$

Since the value of our test statistic is contained inside the rejection region, we reject the null hypothesis in favour of the alternative hypothesis.

(b) Find the (approximate) p-value. Hint: use the fact that, in view of the CLT, when $p = 0.3$,

$$T(X_1, X_2, \dots, X_n) \overset{\text{approx}}{\sim} N(0,1),$$

so that the p-value is given by

$$\mathbf{P}_{p=0.3}\left(T(X_1, X_2, \dots, X_n) \geq T(x_1, x_2, \dots, x_n)\right) \approx \mathbf{P}\left(N(0,1) \geq T(x_1, x_2, \dots, x_n)\right),$$

where $T(x_1, x_2, \dots, x_n)$ is the observed value of the test statistic.

As this is an upper-tailed test, the p-value is found as the area to the right of our observed test statistic using the standard normal distribution (by CLT).

```r
pnorm(statistic, lower.tail=FALSE)
```

```
## [1] 0.008413705
```

As the p-value is less than 0.05, we once again reject the null hypothesis in favour of the alternative hypothesis.

On the interpretation of p-values: a p-value is the probability of observing a statistic that is equally or more extreme than the current observed statistic under the assumption that the null hypothesis is true. In the case of the upper-tailed test (above), the (approximate) p-value was obtained as:

$$p\text{-value} = \mathbf{P}\left(Z \geq 2.390 \,|\, H_0 \text{ is true}\right),$$

since *more extreme* would imply greater values of $\hat{p}$, leading to greater values of the test statistic. In general, a small p-value means that the probability of observing a statistic that is equally or more extreme than the currently observed one, under the assumption that the null hypothesis is true, is very small. As such, one can think of a small p-value as evidence against the null hypothesis.

When the p-value is smaller than α, the null hypothesis is rejected in favour of the alternative hypothesis. The conclusion from using a p-value is always identical to the conclusion made using the rejection region method.