Tutorial 8 Solutions

Question 1

Specimens of milk from a number of dairies in three different districts were analysed, and the concentration of the radioactive isotope strontium-90 was measured in each specimen. Suppose that specimens were obtained from 4 dairies in the first district, 6 dairies in the second district, and 3 dairies in the third district, measured in picocuries per litre.

District (i)	n_i	Concentration
1	4	6.4
		5.8
		6.5
		7.7
2	6	7.1
		9.9
		11.2
		10.5
		6.5
		8.8
3	3	9.5
		9.0
		12.1

(a) Assuming that the variance of the concentration of strontium-90 is the same for the dairies in all three districts, determine the maximum likelihood estimate of the mean concentration in each of the districts and the maximum likelihood estimate of the common variance.

It can be found that the group means are:

$$\overline{Y}_{1\bullet} = 6.6, \quad \overline{Y}_{2\bullet} = 9.0, \quad \overline{Y}_{3\bullet} = 10.2.$$

It was shown in class that in the case of a one-way layout with k independent samples of sizes $n_1, \ldots, n_k$, where $n_1 + \ldots + n_k = n$ is the total number of observations, the sample mean of the i^{th} group, $\overline{Y}_{i\bullet}$, is the MLE for μ_i, for $i = 1, \ldots, k$, where μ_i in the context of this problem represents the true mean concentration of strontium-90 for district i. In addition, it was shown that the MLE for σ^2 was given by:

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \overline{Y}_{i\bullet})^2.$$

Next, it can be found that:

$$\sum_{j=1}^{4}(Y_{1j} - \overline{Y}_{1\bullet})^2 = 1.90$$

$$\sum_{j=1}^{6}(Y_{2j} - \overline{Y}_{2\bullet})^2 = 17.8$$

$$\sum_{j=1}^{3}(Y_{3j} - \overline{Y}_{3\bullet})^2 = 5.54$$

Plugging these quantities into our formula for the MLE of $\hat{\sigma}^2$, we obtain:

$$\hat{\sigma}^2 = \frac{1.90 + 17.8 + 5.54}{13} = 1.942.$$

(b) Test the hypothesis that the three districts have identical concentrations of strontium-90.

The hypothesis we wish to test is:

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu \quad \text{vs} \quad H_1 : \text{At least one } \mu_i \neq \mu$$

The test statistic is computed as:

$$F = \frac{\text{SST}/(k-1)}{\text{SSE}/(n-k)} \sim F_{k-1,\,n-k},$$

under the null hypothesis. The required quantities are computed using the formulas given on pages 668-669 of the textbook:

$$\overline{Y} = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n_i} Y_{ij} = 8.538 \qquad \text{(The grand mean)}$$

$$\text{SST} = \sum_{i=1}^{k} n_i(\overline{Y}_{i\bullet} - \overline{Y})^2 = 24.591$$

$$\text{SSE} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \overline{Y}_{i\bullet})^2 = 25.24 \qquad \text{(Using values from (a))}$$

Thus the value of our test statistic is:

$$F = \frac{24.591/2}{25.24/10} = 4.871.$$

Using Table 7, the p-value is between 0.025 and 0.05. Therefore, at the 5% significance level, we would reject the null hypothesis. However, if we were using a significance level of 2.5%, we would fail to reject the null hypothesis.

If we were to reject the null hypothesis at the 5% significance level, we would conclude that at least one pair of districts has a different mean concentration of strontium-90 in their milk.

Question 2

Show that in a one-way layout, the following statistic is an unbiased estimator of σ^2:

$$\frac{1}{n-k}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij}-\overline{Y}_{i\bullet})^2$$

where

$$\overline{Y}_{i\bullet} = \frac{1}{n_i}\sum_{j=1}^{n_i}Y_{ij}.$$

By assumption, we have k independent samples, $\mathbf{Y}_{i\bullet}$, $i = 1, \ldots, k$, where each sample arises from a normal distribution with unknown mean μ_i and unknown common variance σ^2.

Recall that for a random sample on size n from a $N(\mu, \sigma^2)$,

$$\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{\sigma^2} \sim \chi^2_{n-1}.$$

It follows then that

$$\frac{\sum_{j=1}^{n_i}(Y_{ij}-\overline{Y}_{i\bullet})^2}{\sigma^2} \sim \chi^2_{n_i-1}, \quad i = 1, \ldots, k.$$

As such,

$$\mathbf{E}\left(\frac{\sum_{j=1}^{n_i}(Y_{ij}-\overline{Y}_{i\bullet})^2}{\sigma^2}\right) = n_i - 1 \quad \Longleftrightarrow \quad \mathbf{E}\left(\sum_{j=1}^{n_i}(Y_{ij}-\overline{Y}_{i\bullet})^2\right) = \sigma^2(n_i-1),$$

for $i = 1, \ldots, k$, as σ^2 is simply a constant.

Taking the expectation of the above statistic:

$$\begin{aligned}
\mathbf{E}\left(\frac{1}{n-k}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij}-\overline{Y}_{i\bullet})^2\right) &= \frac{1}{n-k}\sum_{i=1}^{k}\mathbf{E}\left(\sum_{j=1}^{n_i}(Y_{ij}-\overline{Y}_{i\bullet})^2\right) \\
&= \frac{1}{n-k}\sum_{i=1}^{k}\sigma^2(n_i-1) \\
&= \frac{\sigma^2}{n-k}\sum_{i=1}^{k}(n_i-1) \\
&= \frac{\sigma^2}{n-k}\left(\sum_{i=1}^{k}n_i - k\right)
\end{aligned}$$

$$= \frac{\sigma^2}{n-k}(n-k)$$

$$= \sigma^2$$

In conclusion, the statistic given above is an unbiased estimator of σ^2.

Question 3

(13.8) In a study of starting salaries for assistant professors, five male assistant professors at each of three types of doctoral-granting institutions were randomly polled and their starting salaries were recorded under the condition of anonymity. The results of the survey (measured in $1000) are given in the following table.

Public Universities	Private-Independent	Church-Affiliated
49.3	81.8	66.9
49.9	71.2	57.3
48.5	62.9	57.7
68.5	69.0	46.2
54.0	69.0	52.2

(a) **R** Is there sufficient evidence to indicate a difference in the average starting salaries of assistant professors at the three types of doctoral-granting institutions? Use the rejection region method.

We begin by reading the data into **R**. I will be entering the data by column – I first enter the public salaries, then the private salaries, then the church salaries.

```
salary <- data.frame(
  starting_salary = c(49.3, 49.9, 48.5, 68.5, 54.0, 81.8, 71.2,
                      62.9, 69.0, 69.0, 66.9, 57.3, 57.7, 46.2, 52.2),
  school_type = rep(c("Public", "Private", "Church"), times=c(5, 5, 5))
)

head(salary)
```

```
##   starting_salary school_type
## 1            49.3      Public
## 2            49.9      Public
## 3            48.5      Public
## 4            68.5      Public
## 5            54.0      Public
## 6            81.8     Private
```

Let the subscripts 1 represent church, 2 represent private, and 3 represent public. This is due to the fact that when we call the `aov()` function later, string variables are converted to factors, and string factor levels in **R** are ordered alphabetically by default. The hypotheses that we are interested in testing are:

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu \quad \text{vs} \quad H_1 : \text{At least one } \mu_i \neq \mu$$

Assuming that the required conditions to perform this test hold, we can use the following code to get the F table:

```
aov(starting_salary ~ school_type, data=salary) |>
  summary()
```

```
##             Df Sum Sq Mean Sq F value  Pr(>F)   
## school_type  2  835.0   417.5   7.123 0.00913 **
## Residuals   12  703.3    58.6                   
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

From the resulting output, our F-value is 7.123. Note that the numerator and denominator degrees of freedom can be read off the chart directly: the numerator degrees of freedom is 2 and the denominator degrees of freedom is 12. The critical value is found as:

```
qf(0.05, df1=2, df2=12, lower.tail=FALSE)
```

```
## [1] 3.885294
```

As this is always an upper-tailed test, we shall reject the null hypothesis if our observed F-value is greater than the critical value. Since $7.123 > 3.885$, we reject the null hypothesis in favour of the alternative. We conclude that there is sufficient evidence to support the claim that the mean salaries of at least one class of doctoral-granting institutions differs from the others.

(b) Repeat the above using the p-value method.

The table above also gave a p-value that corresponded to the observed F-value and the hypotheses that we were interested in testing. As usual with the p-value method, we reject the null hypothesis if the p-value is less than the level of significance. Since $0.00913 < 0.05$, we reject the null hypothesis once again, and make the same concluding remarks as in (a).

Question 4

(13.10) An incredibly long story that I don't want to re-write.

Method A	Method B	Method C
73	54	79
83	74	95
76	71	87
68		
80		

(a) Do the data provide sufficient evidence to indicate that at least one of the methods of treatment produces a mean student response different from the other methods? Use the rejection region method at the 5% level of significance.

We begin by reading the data into **R**. I will be entering the data by column – I first enter the method A values, then the method B values, then the method C values.

```
hostility <- data.frame(
  response = c(73, 83, 76, 68, 80, 54, 74, 71, 79, 95, 87),
  method = rep(c("A", "B", "C"), times=c(5, 3, 3))
)

head(hostility)
```

```
##   response method
## 1       73      A
## 2       83      A
## 3       76      A
## 4       68      A
## 5       80      A
## 6       54      B
```

Let the subscripts 1 represent method A, 2 represent method B, and 3 represent method C. Once again, when we call the `aov()` function later, string variables are converted to factors, and string factor levels in **R** are ordered alphabetically by default. The hypotheses that we are interested in testing are:

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu \quad \text{vs} \quad H_1 : \text{At least one } \mu_i \neq \mu$$

Assuming that the required conditions to perform this test hold, we can use the following code to get the F table:

```
aov(response ~ method, data=hostility) |>
  summary()
```

```
##             Df Sum Sq Mean Sq F value Pr(>F)
## method       2  641.9   320.9   5.149 0.0365 *
## Residuals    8  498.7    62.3
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

From the resulting output, our F-value is 5.149. Once again, note that the numerator and denominator degrees of freedom can be read off the chart directly: the numerator degrees of freedom is 2 and the denominator degrees of freedom is 8. The critical value is found as:

```
qf(0.05, df1=2, df2=8, lower.tail=FALSE)
```

```
## [1] 4.45897
```

As this is always an upper-tailed test, we shall reject the null hypothesis if our observed F-value is greater than the critical value. Since $5.149 > 4.459$, we reject the null hypothesis in favour of the alternative. We conclude that there is sufficient evidence to support the claim that at least one treatment method has a different mean response compared to the other treatment methods.

(b) Repeat the above using the p-value method.

The table above also gave a p-value that corresponded to the observed F-value and the hypotheses that we were interested in testing. As usual with the p-value method, we reject the null hypothesis if the p-value is less than the level of significance. Since $0.0365 < 0.05$, we reject the null hypothesis once again, and make the same concluding remarks as in (a).