# Stat 2605 Tutorial 2

September 27, 2022

1. Two different numbers are selected at random from the 10 integers: 1, 2, ..., 10. What is the probability of the event $E$ of getting an even sum?

   To get an even sum, there are two cases to consider:

   - Case 1: even + even

     For the first even number, we have five options (2, 4, 6, 8, 10). Once the first even number has been chosen, there remains four even numbers to choose from. The number of ways to obtain two distinct even numbers is $5 * 4 = 20$.

   - Case 2: odd + odd

     For the first odd number, we have five options (1, 3, 5, 7, 9). As before, once the first odd number has been chosen, there remains four odd numbers to choose from. The number of ways to obtain two distinct odd numbers is also $5 * 4 = 20$.

   Therefore, the number of *favourable* outcomes is $20 + 20 = 40$. The total number of outcomes, i.e. the number of ways we can choose two different numbers is $10 * 9 = 90$. Therefore, the probability of getting an even sum is

   $$\mathbf{P}(E) = \frac{20+20}{90} = \frac{4}{9}.$$

2. An urn contains three red balls and two white balls. Each time a ball is randomly selected and taken out without replacement, two balls of the opposite color are put back into the urn. Find the probability of the event $A$ that the person takes out a red ball first and a white ball next.

   Let $R_1$ be the event that a red ball is drawn first, and let $W_2$ be the event that a white ball is drawn second, such that $A = R_1 \cap W_2$.

   Note that when we begin this procedure, the probability of drawing a red ball (first) is $3/5$.

   Once the first red ball is drawn, we must add two white balls into the urn. At this point, there are two red balls and *four* white balls. The probability of drawing a white ball now is $4/6$.

   Using the multiplication rule, the probability of drawing a red ball first and a white ball second would be

   $$\frac{3}{5} * \frac{4}{6} = \frac{12}{30} = \frac{2}{5}.$$

   We can formalise our intuitive solution above using the conditional probability formula:

   $$\mathbf{P}(A) = \mathbf{P}(R_1 \cap W_2)$$

$$
\begin{aligned}
&= \frac{\mathbf{P}(R_1 \cap W_2)}{\mathbf{P}(R_1)} \cdot \mathbf{P}(R_1) \\
&= \mathbf{P}(W_2 \mid R_1) \cdot \mathbf{P}(R_1) \\
&= \frac{4}{6} \cdot \frac{3}{5} \\
&= \frac{2}{5}.
\end{aligned}
$$

3. Suppose that $A_1$, $A_2$, and $A_3$ are independent events with $\mathbf{P}(A_1) = 0.2$, $\mathbf{P}(A_2) = 0.4$, and $\mathbf{P}(A_3) = 0.7$. Find $\mathbf{P}(A_1 \cap A_2^c \cap A_3^c)$.

   By the properties of independent events,

$$
\begin{aligned}
\mathbf{P}(A_1 \cap A_2^c \cap A_3^c) &= \mathbf{P}(A_1) \cdot \mathbf{P}(A_2^c) \cdot \mathbf{P}(A_3^c) \\
&= \mathbf{P}(A_1) \cdot (1 - \mathbf{P}(A_2)) \cdot (1 - \mathbf{P}(A_3)) \\
&= 0.2 \cdot (1 - 0.4) \cdot (1 - 0.7) \\
&= 0.2 \cdot 0.6 \cdot 0.3 \\
&= 0.036.
\end{aligned}
$$

4. (a) A box contains four white balls and six red balls. Four balls are selected at random with replacement. Find the probability of getting two white and two red balls.

   With replacement, the probability of drawing a white ball is $4/10$ and the probability of drawing a red ball is $6/10$.

   To get our desired probability, we also need to consider that the specific ordering of the balls does not matter – only that we have exactly two white and two red. The number of ways that we can get two white and two red balls is obtained by thinking of the drawn balls by their "positions". Of these four potential positions, we shall fix two of them to be white. Note that by default, this means that the remaining two positions will be "not white", e.g. red. Then the number of ways to do this is $\binom{4}{2} = 6$.

   Putting it all together, the probability of selecting four balls with replacement such that there are two white and two red balls is

$$
\binom{4}{2}(0.4)^2(0.6)^2 = 0.3456.
$$

   This formulation is related to the *binomial distribution*.

   (b) How does the answer to (a) change if the balls are drawn without replacement?

   If we sample without replacement, the number of favourable outcomes is $\binom{4}{2} \cdot \binom{6}{2} = 90$.

   The total possible number of outcomes is $\binom{10}{4} = 210$.

   The probability of selecting four balls without replacement such that there are two white and two red balls is

$$\frac{\binom{4}{2}\cdot\binom{6}{2}}{\binom{10}{4}} = \frac{90}{210} = \frac{3}{7} = 0.4286.$$

This formulation is related to the *hypergeometric distribution*.

5. A box contains four good items and three defective items. Three items are selected at random without replacement. Let $X$ be the number of good items. Find the pmf and cdf of $X$.

If $X$ represents the number of good items in a sample of size three, then $X$ can only take on the values 0, 1, 2, and 3. Note that although our box contains four good items, we can never pull four good items since our sample has a size of three!

The probabilities that make up the pmf will be obtained in a manner similar to that of 4(b). We can obtain our pmf by beginning to list off the individual cases.

- Case 1: In a sample of size three (without replacement), we obtain zero good items. The probability of this occurring is:

$$\frac{\binom{4}{0}\cdot\binom{3}{3}}{\binom{7}{3}} = \frac{1}{35}.$$

- Case 2: In a sample of size three (without replacement), we obtain one good item. The probability of this occurring is:

$$\frac{\binom{4}{1}\cdot\binom{3}{2}}{\binom{7}{3}} = \frac{12}{35}.$$

- Case 3: In a sample of size three (without replacement), we obtain two good items. The probability of this occurring is:

$$\frac{\binom{4}{2}\cdot\binom{3}{1}}{\binom{7}{3}} = \frac{18}{35}.$$

- Case 4: In a sample of size three (without replacement), we obtain two good items. The probability of this occurring is:

$$\frac{\binom{4}{3}\cdot\binom{3}{0}}{\binom{7}{3}} = \frac{4}{35}.$$

Notice that when we know how many good items are in the sample, we automatically know how many defective items are in the sample (because our sample *always* needs to be of size three). Let us formalize the results above.

In a sample of size three (without replacement), let $x$ be the number of good items. Then the pmf of $X$ is given by:

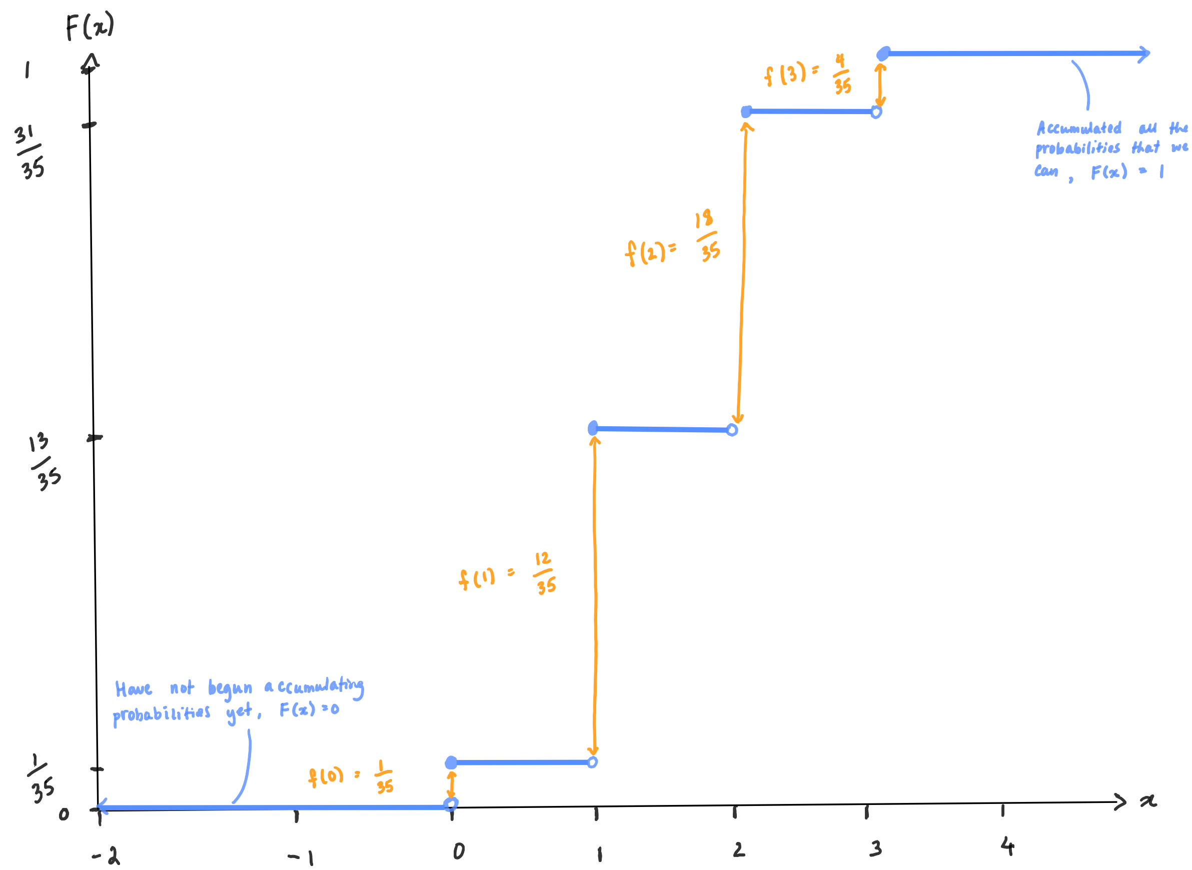
$$\begin{aligned} f(x) &= \mathbf{P}(X = x) \\ &= \begin{cases} \dfrac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}} & x = 0,1,2,3 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1/35 & x = 0 \\ 12/35 & x = 1 \\ 18/35 & x = 2 \\ 4/35 & x = 3 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We can verify that this is a valid pmf since all the probabilities sum to one. Note that we must always include the "0 otherwise" case since probability distribution functions are defined for all real inputs. Omitting this would make it appear as though your probability distribution function is undefined at points other than those listed, which is untrue.

The cmf is obtained by cumulatively summing our probabilities at each point in our support. We must also keep in mind that the cmf has a value of zero when $x < 0$ (0 is the smallest value in our support) because we do not have any non-zero probabilities to sum. In addition, the cmf has a value of one when $x \geq 3$ (3 is the largest value in our support) because we have summed up all the probabilities at this point.

$$\begin{aligned} F(x) &= \mathbf{P}(X \leq x) \\ &= \begin{cases} 0 & x < 0 \\ \frac{1}{35} & 0 \leq x < 1 \\ \frac{1+12}{35} & 1 \leq x < 2 \\ \frac{1+12+18}{35} & 2 \leq x < 3 \\ \frac{1+12+18+4}{35} & x \geq 3 \end{cases} \\ &= \begin{cases} 0 & x < 0 \\ 1/35 & 0 \leq x < 1 \\ 13/35 & 1 \leq x < 2 \\ 31/35 & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases} \end{aligned}$$

We can draw a graph of $F(x)$ to see what it looks like and also its relationship to the values of $f(x)$.




Note: plot is not drawn to scale.