Stat 2605 Tutorial 3

October 11, 2022

1. Suppose X has pmf given by

$$f(x) = \begin{cases} 0.2 & x = 1 \\ 0.3 & x = 2 \\ 0.1 & x = 3 \\ 0.4 & x = 5 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Calculate $\mathbf{P}(X \leq 3)$.

$$\begin{aligned} \mathbf{P}(X \leq 3) &= \mathbf{P}(X = 1) + \mathbf{P}(X = 2) + \mathbf{P}(X = 3) \\ &= 0.2 + 0.3 + 0.1 \\ &= 0.6 \end{aligned}$$

   (b) Calculate $\mathbf{E}\left(X^2\right)$.

$$\begin{aligned} \mathbf{E}\left(X^2\right) &= \sum_{x \in \mathcal{S}} x^2 \cdot f(x) \\ &= 1^2 \cdot 0.2 + 2^2 \cdot 0.3 + 3^2 \cdot 0.1 + 5^2 \cdot 0.4 \\ &= 12.3 \end{aligned}$$

   (c) Sketch the cdf.

   See last page.

2. Let X be the outcome when a fair die is tossed.

   (a) Calculate $\mathbf{Var}(X)$.

   $\mathbf{Var}(X) = \mathbf{E}\left(X^2\right) - (\mathbf{E}(X))^2$. Therefore, in order to calculate $\mathbf{Var}(X)$, we must first calculate $\mathbf{E}(X)$ and $\mathbf{E}\left(X^2\right)$.

$$\begin{aligned} \mathbf{E}(X) &= \sum_{x \in \mathcal{S}} x \cdot f(x) \\ &= \sum_{x=1}^{6} x \cdot \frac{1}{6} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6}(1+2+3+4+5+6) \\
&= \frac{21}{6} \\
&= \frac{7}{2}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}\left(X^2\right) &= \sum_{x \in \mathcal{S}} x^2 \cdot f(x) \\
&= \sum_{x=1}^{6} x^2 \cdot \frac{1}{6} \\
&= \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) \\
&= \frac{91}{6}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Var}\,(X) &= \mathbf{E}\left(X^2\right) - \left(\mathbf{E}\,(X)\right)^2 \\
&= \frac{91}{6} - \left(\frac{7}{2}\right)^2 \\
&= \frac{35}{12} \\
&= 2.91\overline{66}
\end{aligned}
$$

(b) Calculate $\mathbf{SD}\,(X)$.

$$
\mathbf{SD}\,(X) = \sqrt{\mathbf{Var}\,(X)} = \sqrt{\frac{35}{12}} \approx 1.7078
$$

3. A fair die is tossed 10 times. Let Y denote the number of times a one occurs.

In an experiment consisting of 10 tosses, if we consider the rolling of a one as a "success", and the rolling of any other number as a a "fail", then we can say that

$$
Y \sim \text{Binomial}(n = 10,\ p = 1/6).
$$

(a) Calculate the mean of Y.

If Y has a binomial distribution, then its mean is np.

$$
\mathbf{E}\,(Y) = np = 10 \cdot \frac{1}{6} = \frac{5}{3}
$$

(b) Calculate the variance of Y.

If Y has a binomial distribution, then its variance is npq, where $q = (1-p)$.

$$\mathbf{Var}(Y) = npq = 10 \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{50}{36} = \frac{25}{18}$$

(c) Calculate the probability of observing at least four but no more than eight ones.

The Binomial($n = 10$, $p = 1/6$) pmf is:

$$f(y) = \mathbf{P}(Y = y) = \begin{cases} \binom{10}{y} (1/6)^y (5/6)^{10-y} & y = 0, 1, 2, \ldots, 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathbf{P}(4 \le Y \le 8) &= \sum_{y=4}^{8} \mathbf{P}(Y = y) \\ &= \binom{10}{4} (1/6)^4 (5/6)^6 + \binom{10}{5} (1/6)^5 (5/6)^5 + \binom{10}{6} (1/6)^6 (5/6)^4 \\ &\quad + \binom{10}{7} (1/6)^7 (5/6)^3 + \binom{10}{8} (1/6)^8 (5/6)^2 \\ &\approx 0.069727 \end{aligned}$$

4. An unfair coin where $\mathbf{P}(H) = 0.6$ is repeatedly tossed. Let T be the number of tosses until a head is observed.

Let the event that a head is observed be deemed a "success". If we can denote the probability of success as

$$p = \mathbf{P}(H) = 0.6,$$

then

$$T \sim \text{Geometric}(p = 0.6).$$

(a) Calculate $\mathbf{E}(T)$.

If T has a geometric distribution, then its mean is $1/p$.

$$\mathbf{E}(T) = \frac{1}{p} = \frac{1}{0.6} = \frac{5}{3}$$

(b) Calculate $\mathbf{P}(T \ge 3)$.

The pmf of T is

$$f(t) = \mathbf{P}(T = t) = \begin{cases} (0.4)^{t-1}(0.6) & t \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Using an indirect approach:

$$\begin{aligned} \mathbf{P}(T \geq 3) &= 1 - \mathbf{P}(T < 3) \\ &= 1 - \mathbf{P}(T \leq 2) \\ &= 1 - \sum_{t=1}^{2} \mathbf{P}(T = t) \\ &= 1 - (0.4)^0(0.6) - (0.4)(0.6) \\ &= 0.16 \end{aligned}$$

Using a direct approach:
Let $p = 0.6$ and $q = 0.4$.

$$\begin{aligned} \mathbf{P}(T \geq 3) &= \sum_{t=3}^{\infty} \mathbf{P}(T = t) \\ &= q^2p + q^3p + q^4p + q^5p + \dots \\ &= q^2p(1 + q + q^2 + q^3 + \dots) \\ &= q^2p \sum_{i=1}^{\infty} q^{i-1} \\ &= \frac{q^2p}{1-q} \\ &= \frac{(0.4)^2(0.6)}{1-0.4} \\ &= 0.16 \end{aligned}$$

5. When parts from an assembly line are inspected, 2% of them are found to be defective. Suppose that 100 units are tested. Find the probability that three units are defective by using a Poisson approximation.

Let X be the number of units that are defective in a sample of 100 units. The true distribution of X is Binomial($n = 100$, $p = 0.02$). We can make a Poisson approximation of the binomial distribution provided that n is large and p is small. As a rule of thumb, we usually want $n > 50$, and $np < 5$. In this case, since $np = 100 * 0.02 = 2$, we proceed with the Poisson approximation such that

$$X \dot{\sim} \text{Poisson}(\lambda = np = 2).$$

Recall that the Poisson pmf is given by

$$f(x) = \mathbf{P}(X = x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\mathbf{P}(X = 3) \approx \frac{e^{-2}\, 2^3}{3!} \approx 0.180447.$$

The true value of $\mathbf{P}(X = 3)$ using the binomial pmf is approximately 0.182276.

6. Suppose X and Y are two random variables with

$$\mathbf{E}(X) = 1 \qquad \mathbf{E}\left(X^2\right) = 10$$
$$\mathbf{E}(Y) = -2$$

Let $Z = X + Y$ and $W = -2X$.

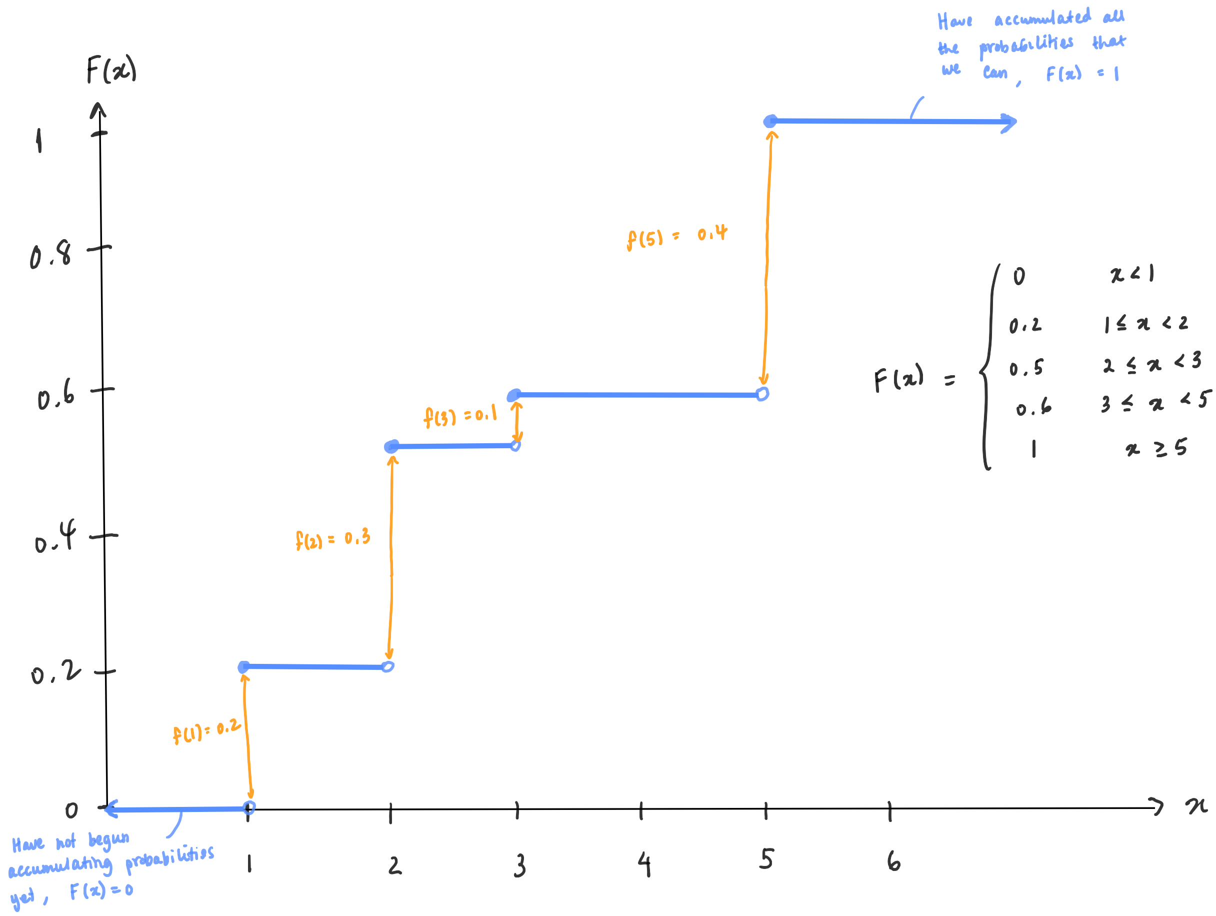
(a) Calculate $\mathbf{E}(Z)$.

$$\mathbf{E}(Z) = \mathbf{E}(X + Y) = \mathbf{E}(X) + \mathbf{E}(Y) = 1 - 2 = -1$$

(b) Calculate $\mathbf{SD}(W)$.

$$\begin{aligned} \mathbf{Var}(W) &= \mathbf{Var}(-2X) \\ &= (-2)^2\mathbf{Var}(X) \\ &= 4\left(\mathbf{E}\left(X^2\right) - (\mathbf{E}(X))^2\right) \\ &= 4(10 - 1^2) \\ &= 36 \end{aligned}$$

$$\mathbf{SD}(W) = \sqrt{\mathbf{Var}(W)} = \sqrt{36} = 6$$



Note: plot not drawn to scale.