Stat 2605 Tutorial 4

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1. Suppose X has pdf given by

$$f(x) = \begin{cases} c(2x+1) & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find c.

In order for $f(x)$ to be a valid density, we seek a value c such that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= c\int_{1}^{2} 2x+1\,dx \\ &= c(x^2+x)\Big|_{x=1}^{x=2} \\ &= c\left((2^2+2)-(1^2+1)\right) \\ &= c\cdot 4 \end{aligned}$$

$$c\cdot 4 = 1 \quad \iff \quad c = \frac{1}{4}$$

Updating our pdf above, we have:

$$f(x) = \begin{cases} \frac{1}{4}(2x+1) & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) Find $\mathbf{P}(X > 3/2)$.

Indirect method

$$\begin{aligned} \mathbf{P}(X \le 3/2) &= \int_{1}^{3/2} \frac{1}{4}(2x+1)\,dx \\ &= \frac{1}{4}(x^2+x)\Big|_{x=1}^{x=3/2} \\ &= \frac{1}{4}\left(\left(\left(\frac{3}{2}\right)^2+\frac{3}{2}\right) - (1^2+1)\right) \end{aligned}$$

$$= \frac{1}{4}\left(\frac{15}{4} - \frac{8}{4}\right)$$

$$= \frac{1}{4} \cdot \frac{7}{4}$$

$$= \frac{7}{16}$$

$$\mathbf{P}\left(X > 3/2\right) = 1 - \mathbf{P}\left(X \le 3/2\right) = 1 - \frac{7}{16} = \frac{9}{16}$$

Direct method

$$\mathbf{P}\left(X > 3/2\right) = \int_{3/2}^{2} \frac{1}{4}(2x+1)\,dx$$

$$= \frac{1}{4}(x^2 + x)\Big|_{x=3/2}^{x=2}$$

$$= \frac{1}{4}\left((2^2+2) - \left(\left(\frac{3}{2}\right)^2 + \frac{3}{2}\right)\right)$$

$$= \frac{1}{4}\left(6 - \frac{15}{4}\right)$$

$$= \frac{1}{4}\left(\frac{24}{4} - \frac{15}{4}\right)$$

$$= \frac{1}{4} \cdot \frac{9}{4}$$

$$= \frac{9}{16}$$

(c) Find $\mathbf{E}\left(X\right)$.

$$\mathbf{E}\left(X\right) = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

$$= \int_{1}^{2} x \cdot \frac{1}{4}(2x+1)\,dx$$

$$= \frac{1}{4}\int_{1}^{2} 2x^2 + x\,dx$$

$$= \frac{1}{4}\left(\frac{2}{3}x^3 + \frac{1}{2}x^2\right)\Big|_{x=1}^{x=2}$$

$$= \frac{1}{4}\left(\left(\frac{16}{3}+2\right) - \left(\frac{2}{3}+\frac{1}{2}\right)\right)$$

$$= \frac{37}{24}$$

(d) Find **Var** (X).

$$\mathbf{E}\left(X^2\right) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx$$

$$= \int_{1}^{2} x^2 \cdot \frac{1}{4}(2x+1)\,dx$$

$$= \frac{1}{4}\int_{1}^{2} 2x^3 + x^2\,dx$$

$$= \frac{1}{4}\left(\frac{1}{2}x^4 + \frac{1}{3}x^3\right)\Big|_{x=1}^{x=2}$$

$$= \frac{1}{4}\left(\left(\frac{1}{2}(2)^4 + \frac{1}{3}(2)^3\right) - \left(\frac{1}{2}(1)^4 + \frac{1}{3}(1)^3\right)\right)$$

$$= \frac{1}{4}\left(\left(8+\frac{8}{3}\right) - \left(\frac{1}{2}+\frac{1}{3}\right)\right)$$

$$= \frac{59}{24}$$

$$\mathbf{Var}(X) = \mathbf{E}\left(X^2\right) - (\mathbf{E}(X))^2 = \frac{59}{24} - \left(\frac{37}{24}\right)^2 = \frac{47}{576}$$

2. Suppose X has pdf given by

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = X^3 + X$.

(a) Find the cdf of X.

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)\,dt \\
&= \int_{0}^{x} 3t^2\,dt \\
&= t^3\Big|_{t=0}^{t=x} \\
&= x^3 - 0^3 \\
&= x^3
\end{aligned}
$$

$$
F(x) = \begin{cases} 0 & x \le 0 \\ x^3 & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}
$$

(b) Find $\mathbf{E}(Y)$.

Think of Y as a function of X, $g(X)$.

$$
\begin{aligned}
\mathbf{E}(Y) &= \mathbf{E}(g(X)) \\
&= \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx \\
&= \int_{0}^{1} (x^3 + x) \cdot 3x^2\,dx \\
&= 3\int_{0}^{1} x^5 + x^3\,dx \\
&= 3\left(\frac{1}{6}x^6 + \frac{1}{4}x^4\right)\Bigg|_{x=0}^{x=1} \\
&= 3\left(\frac{1}{6} + \frac{1}{4}\right) \\
&= \frac{5}{4}
\end{aligned}
$$

3. Suppose X has an exponential distribution with pdf given by

$$
f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}
$$

Find $\mathbf{E}(X)$.

$$\mathbf{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$= \int_{0}^{\infty} x \cdot 2e^{-2x}\,dx$$

$$= 2\int_{0}^{\infty} xe^{-2x}\,dx$$

$$f(x) = x \qquad g(x) = -\tfrac{1}{2}e^{-2x}$$

$$f'(x) = 1 \qquad g'(x) = e^{-2x}$$

$$= 2\left[-\frac{1}{2}xe^{-2x}\Big|_{x=0}^{x=\infty} - 1 \cdot \left(-\frac{1}{2}\int_{0}^{\infty} e^{-2x}\,dx \right) \right]$$

$$= 2\left[(0-0) + \frac{1}{2}\left(-\frac{1}{2}e^{-2x} \right)\Big|_{x=0}^{x=\infty} \right]$$

$$= 2\left(-\frac{1}{4}(0-1) \right)$$

$$= 2 \cdot \frac{1}{4}$$

$$= \frac{1}{2}$$