Stat 2605 Tutorial 5

November 1, 2022

1. Suppose that X has an Exp($\lambda = 2$) distribution with pdf given by:

$$f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate $\mathbf{P}(X > 2)$.

$$\mathbf{P}(X > 2) = \int_2^\infty f(x)\,dx$$

$$= \int_2^\infty 2e^{-2x}\,dx$$

$$d(-2x) = -2dx \iff -d(-2x) = 2dx$$

$$= -\int_2^\infty e^{-2x}\,d(-2x)$$

$$= -\,e^{-2x}\Big|_{x=2}^{x=\infty}$$

$$= -(0 - e^{-4})$$

$$= e^{-4}$$

(b) If $X \sim$ Exp($\lambda = 2$), what is $\mathbf{E}(X)$ and $\mathbf{Var}(X)$? Use this to calculate $\mathbf{E}\left(X^2\right)$.

We saw last time (Tutorial 4, Question 3) that if $X \sim$ Exp($\lambda = 2$) then

$$\mathbf{E}(X) = \frac{1}{\lambda} = \frac{1}{2}.$$

It can also be shown that

$$\mathbf{Var}(X) = \frac{1}{\lambda^2} = \frac{1}{2^2} = \frac{1}{4}.$$

To find $\mathbf{E}\left(X^2\right)$ with the above given information, we simply need to rearrange the usual variance formula.

$$\mathbf{Var}(X) = \mathbf{E}\left(X^2\right) - (\mathbf{E}(X))^2 \iff \mathbf{E}\left(X^2\right) = \mathbf{Var}(X) + (\mathbf{E}(X))^2$$

It follows that

$$\mathbf{E}\left(X^2\right) = \frac{1}{4} + \left(\frac{1}{2}\right)^2 = \frac{1}{2}.$$

2. Suppose $X \sim N(\mu = 4, \sigma^2 = 3^2)$. Calculate $\mathbf{P}(2 < X < 5)$.

Recall that if $X \sim N(\mu, \sigma^2)$ then

$$Z := \frac{X - \mu}{\sigma} \sim N(0, 1).$$

This property is extremely convenient for us because it means that we can use a single table of probabilities for all normal distributions, rather than using a specific probability table for each normal distribution we come across!

$$\begin{aligned}
\mathbf{P}(2 < X < 5) &= \mathbf{P}\left(\frac{2-4}{3} < \frac{X-4}{3} < \frac{5-4}{3}\right) \\
&= \mathbf{P}\left(-\frac{2}{3} < Z < \frac{1}{3}\right) \\
&= \mathbf{P}\left(Z < \frac{1}{3}\right) - \mathbf{P}\left(Z < -\frac{2}{3}\right) \qquad (2.1) \\
&= \mathbf{P}\left(Z < \frac{1}{3}\right) - \mathbf{P}\left(Z > \frac{2}{3}\right) \qquad (2.2) \\
&= \mathbf{P}\left(Z < \frac{1}{3}\right) - \left(1 - \mathbf{P}\left(Z < \frac{2}{3}\right)\right) \qquad (2.3) \\
&= \mathbf{\Phi}(0.33) - (1 - \mathbf{\Phi}(0.67)) \\
&= 0.6293 - (1 - 0.7486) \\
&= 0.3779
\end{aligned}$$

Notes:

(2.1) For a continuous random variable X,

$$\mathbf{P}(a < X < b) = \mathbf{P}(X < b) - \mathbf{P}(X < a).$$

(2.2) Since you will only be given the normal probability table for values of $z \geq 0$, we use the symmetry property of the normal distribution:

$$\mathbf{P}(Z < -z) = \mathbf{P}(Z > z).$$

(2.3) Since the normal probability table gives probabilities of the form $\mathbf{P}(Z < z)$, we use the complement rule:

$$\mathbf{P}(Z > z) = 1 - \mathbf{P}(Z < z)$$

3. Suppose X has pdf given by:

$$f(x) = \begin{cases} \dfrac{2(x+1)}{3} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = e^X + 1$. Find the pdf of Y.

Let $g(X) = Y = e^X + 1$.

We start by solving for X to obtain the inverse of this transformation.

$$\begin{aligned} & e^X + 1 && = Y \\ \iff \quad & e^X && = Y - 1 \\ \iff \quad & X && = \log(Y - 1) \end{aligned}$$

Thus, we have:

$$g(X) = Y = e^X + 1$$

$$g^{-1}(Y) = X = \log(Y - 1)$$

Next, we will need to find $\left|\dfrac{dx}{dy}\right| = \left|\dfrac{d\,g^{-1}(y)}{dy}\right|$.

$$\left|\frac{d\,g^{-1}(y)}{dy}\right| = \left|\frac{d}{dy}\log(y-1)\right| = \left|\frac{1}{y-1}\right|$$

Now, note the new support of Y:

$$\begin{aligned} & 0 < x < 1 \\ \iff \quad & e^0 < e^x < e^1 \\ \iff \quad & e^0 + 1 < e^x + 1 < e^1 + 1 \\ \iff \quad & 2 < y < e + 1 \end{aligned}$$

As such, $\left|\dfrac{d\,g^{-1}(y)}{dy}\right| = \left|\dfrac{1}{y-1}\right|$ is always positive and the absolute values can be dropped.

Finally, the pdf of Y is given by:

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \cdot \left|\frac{d\,g^{-1}(y)}{dy}\right| \\ &= f_X(\log(y-1)) \cdot \frac{1}{y-1} \end{aligned}$$

$$= \frac{2(\log(y-1)+1)}{3(y-1)},$$

for $y \in (2, e+1)$, and zero otherwise.

4. Let X have pdf given by:

$$f(x) = \begin{cases} 4x^3 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = 1/\sqrt{X}$. Find the pdf of Y.

Let $g(X) = Y = 1/\sqrt{X}$. Note that we will ignore the case where $x = 0$ which will result in division by zero.

We start by solving for X to obtain the inverse of this transformation.

$$\begin{aligned} & \frac{1}{\sqrt{X}} && = Y \\ \iff & \sqrt{X} && = \frac{1}{Y} \\ \iff & X && = \frac{1}{Y^2} \end{aligned}$$

Thus, we have:

$$g(X) = Y = \frac{1}{\sqrt{X}}$$

$$g^{-1}(Y) = X = \frac{1}{Y^2}$$

Next, we will find $\left|\frac{dx}{dy}\right| = \left|\frac{d\,g^{-1}(y)}{dy}\right|$.

$$\left|\frac{d\,g^{-1}(y)}{dy}\right| = \left|\frac{d}{dy}y^{-2}\right| = \left|-2y^{-3}\right| = 2\left|y^{-3}\right|$$

Now, we note the new support of Y:

$$\begin{aligned} & 0 < x \le 1 \\ \iff & \sqrt{0} < \sqrt{x} \le \sqrt{1} \\ \iff & \infty > \frac{1}{\sqrt{x}} \ge \frac{1}{\sqrt{1}} \\ \iff & 1 \le y < \infty \end{aligned}$$

As such, $\left|\frac{d\,g^{-1}(y)}{dy}\right| = 2\left|y^{-3}\right|$ is always positive and the absolute values can be dropped.

Finally, the pdf of Y is given by:

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \cdot \left|\frac{d\,g^{-1}(y)}{dy}\right| \\ &= f_X\left(\frac{1}{y^2}\right) \cdot 2y^{-3} \\ &= 4\left(\frac{1}{y^2}\right)^3 \cdot \frac{2}{y^3} \\ &= \frac{8}{y^9}, \end{aligned}$$

for $y \geq 1$, and zero otherwise.

5. Suppose X has pdf given by:

$$f(x) = \begin{cases} \frac{3}{7}x^2 & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Outline the steps required to simulate X by generating a $U \sim \text{Unif}(0, 1)$.

The cdf of X is given by

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\,dt \\ &= \int_{1}^{x} \frac{3}{7}t^2\,dt \\ &= \frac{1}{7}t^3\Big|_{t=1}^{t=x} \\ &= \frac{1}{7}(x^3 - 1) \end{aligned}$$

$$F(x) = \begin{cases} 0 & x \leq 1 \\ \frac{1}{7}(x^3 - 1) & 1 < x < 2 \\ 1 & x \geq 2 \end{cases}$$

If u is the observation generated from the Unif$(0, 1)$, then we set

$$u = \frac{1}{7}(x^3 - 1)$$

and solve for x. This series of steps is fine if we are generating a single observation from the distribution of X.

For larger samples, it may be more convenient to find the *quantile function* which is often the inverse of the cdf. This way, we do not need to repeatedly solve for x.

Let p be some probability such that

$$p = F(x) = \mathbf{P}(X \leq x) = \frac{1}{7}(x^3 - 1).$$

$$\begin{aligned}
&\frac{1}{7}(x^3 - 1) &&= p \\
\iff\quad & x^3 - 1 &&= 7p \\
\iff\quad & x^3 &&= 7p + 1 \\
\iff\quad & x &&= (7p + 1)^{1/3}
\end{aligned}$$

For every value u generated from the Unif$(0, 1)$, we simply substitute it in place of p. Then

$$x = (7u + 1)^{1/3}$$

is a simulated observation from the distribution of X.