Stat 2605 Tutorial 6

November 15, 2022

1. Suppose (X, Y) has joint pmf given by:

$f(x,y)$		**x**		
		1	2	3
	0	0.2	0.1	0.1
y	1	0.1	0.3	0
	2	0	0.1	0.1

and zero otherwise. Find $\mathbf{E}\left(2X^2Y + 1\right)$.

By properties of expectation,

$$\mathbf{E}\left(2X^2Y + 1\right) = 2\mathbf{E}\left(X^2Y\right) + 1$$

As such, we should start by finding $\mathbf{E}\left(X^2Y\right)$.

$$\mathbf{E}\left(X^2Y\right) = \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} x^2 y \cdot f(x,y)$$

(We can ignore all the terms where $y = 0$ and $f(x,y) = 0$ since $x^2y \cdot f(x,y)$ will be zero.)

$$= (1^2 \cdot 1)(0.1) + (2^2 \cdot 1)(0.3) + (2^2 \cdot 2)(0.1) + (3^2 \cdot 2)(0.1)$$

$$= 0.1 + 4(0.3) + 8(0.1) + 18(0.1)$$

$$= 3.9$$

Finally,

$$\mathbf{E}\left(2X^2Y + 1\right) = 2\mathbf{E}\left(X^2Y\right) + 1 = 2(3.9) + 1 = 8.8$$

2. Suppose (X, Y) has joint pmf given by:

$f(x,y)$		**x**		
		2	3	4
	-1	0.1	0.2	0.05
y	0	0.27	0.1	0.15
	1	0	0.03	0.1

and zero otherwise.

(a) Are X and Y independent?

We know that X and Y are independent if $f(x,y) = f_X(x) \cdot f_Y(y) \quad \forall \quad (x,y) \in \mathbb{R}^2$. As such, if there is even one such (x,y) where $f(x,y) \neq f_X(x) \cdot f_Y(y)$, then we can conclude that X and Y are *not* independent.

We start by finding the marginal distributions. To find the marginal distribution of X, $f_X(x)$, we fix $X = x$ and sum over all the values of y, for each $x \in \mathcal{X}$.

$$f_X(x) = \begin{cases} 0.1 + 0.27 + 0 & x = 2 \\ 0.2 + 0.1 + 0.03 & x = 3 \\ 0.05 + 0.15 + 0.1 & x = 4 \\ \quad 0 & \text{otherwise} \end{cases} = \begin{cases} 0.37 & x = 2 \\ 0.33 & x = 3 \\ 0.3 & x = 4 \\ 0 & \text{otherwise} \end{cases}$$

Note that $f_X(x)$ is still a valid probability distribution since its probabilities sum to one.

We find the marginal distribution of Y, $f_Y(y)$, in a similar fashion. We fix $Y = y$ and sum over all the values of x, for each $y \in \mathcal{Y}$.

$$f_Y(y) = \begin{cases} 0.1 + 0.2 + 0.05 & y = -1 \\ 0.27 + 0.1 + 0.15 & y = 0 \\ 0 + 0.03 + 0.1 & y = 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 0.35 & y = -1 \\ 0.52 & y = 0 \\ 0.13 & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

Again, we note that $f_Y(y)$ is still a valid probability distribution since its probabilities sum to one.

There are many pairs (x,y) that we can use to show that $f(x,y) \neq f_X(x) \cdot f_Y(y)$. Let's take $x = 2$ and $y = 0$.

$$f(2,1) = 0$$
$$f_X(2) \cdot f_Y(1) = 0.37 \cdot 0.13 \neq 0$$

As such, X and Y are *not* independent.

(b) Show that $\mathbf{E}(XY) \neq \mathbf{E}(X)\,\mathbf{E}(Y)$.

$$\mathbf{E}(XY) = \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} xy \cdot f(x,y)$$

(We can ignore all the terms where $y = 0$ and $f(x,y) = 0$ since $xy \cdot f(x,y)$ will be zero.)

$$= (2 \cdot (-1))(0.1) + (3 \cdot (-1))(0.2) + (4 \cdot (-1))(0.05)$$
$$+ (3 \cdot 1)(0.03) + (4 \cdot 1)(0.1)$$
$$= (-2)(0.1) + (-3)(0.2) + (-4)(0.05) + (3)(0.03) + (4)(0.1)$$

$$= -0.51$$

$$\mathbf{E}(X) = \sum_{x \in \mathcal{X}} x \cdot f_X(x) = (2)(0.37) + (3)(0.33) + (4)(0.3) = 2.93$$

$$\mathbf{E}(Y) = \sum_{y \in \mathcal{Y}} y \cdot f_Y(y) = (-1)(0.35) + (0)(0.52) + (1)(0.13) = -0.22$$

$$\mathbf{E}(X)\,\mathbf{E}(Y) = (2.93)(-0.22) = -0.6446 \neq \mathbf{E}(XY)$$

3. Suppose (X, Y) has joint pdf given by:

$$f(x,y) = \begin{cases} \dfrac{x+y}{3} & 0 < x < 2, \quad 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.

The marginal pdf of X is found by integrating the joint density over the support of Y (with respect to y).

$$\begin{aligned} f_X(x) &= \int_{\mathcal{Y}} f(x,y)\,dy \\ &= \frac{1}{3}\int_0^1 (x+y)\,dy \\ &= \frac{1}{3}\left(xy + \frac{1}{2}y^2\right)\Bigg|_{y=0}^{y=1} \\ &= \frac{1}{3}\left(x + \frac{1}{2}\right) \\ &= \frac{x}{3} + \frac{1}{6} \end{aligned}$$

$$f_X(x) = \begin{cases} \dfrac{x}{3} + \dfrac{1}{6} & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Notice that $f_X(x)$ is a function *solely* of x and its support *only* depends on x, as we have integrated y out. It should be noted that $f_X(x)$ remains a valid probability distribution. We can verify this by integrating $f_X(x)$ over its support and it should equal one.

The marginal pdf of Y is found by integrating the joint density over the support of X (with respect to x).

$$f_Y(y) = \int_{\mathcal{X}} f(x,y)\,dx$$

$$= \frac{1}{3}\int_0^2 (x+y)\,dx$$

$$= \frac{1}{3}\left(\frac{1}{2}x^2 + xy\right)\Bigg|_{x=0}^{x=2}$$

$$= \frac{1}{3}(2 + 2y)$$

$$= \frac{2}{3}(y + 1)$$

$$f_Y(y) = \begin{cases} \frac{2}{3}(y+1) & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Notice that $f_Y(y)$ is a function *solely* of y and its support *only* depends on y, as we have integrated x out. It should be noted that $f_Y(y)$ remains a valid probability distribution. We can verify this by integrating $f_Y(y)$ over its support and it should equal one.

(b) Are X and Y independent?

By inspection, $f(x,y) \neq f_X(x) \cdot f_Y(y) \quad \forall \quad (x,y) \in \mathbb{R}^2$. Therefore X and Y are *not* independent.

(c) Find the conditional mean $\mathbf{E}(Y \mid X = 1)$.

We first note that the conditional expectation of Y given $X = x$ is defined as:

$$\mathbf{E}(Y \mid X = x) = \int_{-\infty}^{\infty} y \cdot f_{Y|X=x}(y|x)\,dy.$$

As such, we should start by finding $f_{Y|X=x}(y|x)$, which is obtained by using the conditional probability formula.

$$f_{Y|X=x}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{1}{3}(x+y)}{\frac{x}{3} + \frac{1}{6}},$$

for $0 < y < 1$ (with an implicit assumption that $0 < x < 2$ to avoid division by zero), and zero otherwise. Given that $X = 1$, we have

$$f_{Y|X=1}(y|1) = \frac{\frac{1}{3}(1+y)}{\frac{1}{3} + \frac{1}{6}} = \frac{2}{3}(1+y),$$

for $0 < y < 1$, and zero otherwise.

$$\mathbf{E}(Y|X = 1) = \int_{-\infty}^{\infty} y \cdot f_{Y|X=1}(y|1)\,dy$$

$$= \frac{2}{3} \int_0^1 y(1+y)\,dy$$

$$= \frac{2}{3} \int_0^1 (y+y^2)\,dy$$

$$= \frac{2}{3} \left(\frac{1}{2}y^2 + \frac{1}{3}y^3 \right) \Bigg|_{y=0}^{y=1}$$

$$= \frac{2}{3} \cdot \frac{5}{6}$$

$$= \frac{5}{9}$$

As usual, notice that since this was an expectation with respect to y, that there are no more ys in our final answer. In addition, since $x = 1$ was assumed and fixed, our final answer does not depend on x either.

4. Suppose that the distribution of the lifetime of a light bulb has an exponential distribution with a mean of 900 hours. Find the probability that the total lifetime of 20 bulbs exceeds 22000 hours.

Let L represent the lifetime of a light bulb. As given in the question, L will have an exponential distribution whose mean is 900 hours, i.e. $\mathbf{E}(L) = 900$. By the properties of the exponential distribution, we also have that $\mathbf{Var}(L) = (\mathbf{E}(L))^2 = 900^2$.

Let T represent the total lifetime of 20 light bulbs:

$$T = \sum_{i=1}^{20} L_i$$

Then:

$$\begin{aligned}
\mathbf{E}(T) &= \mathbf{E}\left(\sum_{i=1}^{20} L_i\right) && \text{(By definition of } T\text{)} \\
&= \mathbf{E}(L_1) + \mathbf{E}(L_2) + \ldots + \mathbf{E}(L_{20}) && \text{(By properties of expectation operator)} \\
&= 20 \cdot \mathbf{E}(L) && \text{(Each } L_i \text{ is identically distributed)} \\
&= 20 \cdot 900 \\
&= 18000
\end{aligned}$$

$$\begin{aligned}
\mathbf{Var}(T) &= \mathbf{Var}\left(\sum_{i=1}^{20} L_i\right) && \text{(By definition of } T\text{)} \\
&= \mathbf{Var}(L_1) + \mathbf{Var}(L_2) + \ldots + \mathbf{Var}(L_{20}) && \text{(Assume each } L_i \text{ is independent)}
\end{aligned}$$

$$= 20 \cdot \mathbf{Var}(L) \qquad \text{(Each } L_i \text{ is identically distributed)}$$

$$= 20 \cdot 900^2$$

Assuming that a size of 20 is sufficiently large, we apply the central limit theorem (CLT) which says:

$$Z := \frac{T - \mu_T}{\sigma_T} = \frac{T - 18000}{\sqrt{20 \cdot 900^2}} \dot{\sim} N(0, 1).$$

$$\mathbf{P}(T > 22000) = \mathbf{P}\left(\frac{T - 18000}{\sqrt{20 \cdot 900^2}} > \frac{22000 - 18000}{\sqrt{20 \cdot 900^2}}\right)$$

$$\approx \mathbf{P}(Z > 0.9938)$$

$$= 1 - \mathbf{P}(Z \leq 0.9938)$$

$$= 1 - \mathbf{\Phi}(0.9938)$$

$$= 1 - 0.8398$$

$$= 0.1602$$

The probability of the total lifetime of 20 light bulbs exceeding 22000 hours is approximately 0.1602.

5. Suppose the random variable, X, has a Geometric(p) distribution, with pmf given by:

$$f(x) = \begin{cases} (1-p)^{x-1}p & x = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

Find the mgf, $M_X(t)$, and use it to find $\mathbf{E}(X)$.

$$M_X(t) = \mathbf{E}\left(e^{tX}\right)$$

$$= \sum_{x=1}^{\infty} e^{tx} (1-p)^{x-1} p$$

$$= \sum_{x-1=0}^{\infty} e^{tx} (1-p)^{x-1} p$$

Let $j := x - 1$. Then $x = j + 1$.

$$= \sum_{j=0}^{\infty} e^{t(j+1)} (1-p)^{j} p$$

$$= \sum_{j=0}^{\infty} e^{tj} e^{t} (1-p)^{j} p$$

$$= pe^t \sum_{j=0}^{\infty} e^{tj}(1-p)^j$$

$$= pe^t \sum_{j=0}^{\infty} \left(e^t\,(1-p)\right)^j$$

$$= \frac{pe^t}{1-(1-p)e^t},$$

for

$$\begin{aligned} & |e^t(1-p)| < 1 \\ \iff \quad & e^t(1-p) < 1 \qquad \text{(Since both } e^t \text{ and } (1-p) \text{ are always non-negative)} \\ \iff \quad & e^t < \frac{1}{1-p} \\ \iff \quad & t < \log\left(\frac{1}{1-p}\right) \end{aligned}$$

(Alternatively, we can write $t < -\log(1-p)$.)

The first moment, $\mathbf{E}(X)$, is obtained by taking the first derivative of $M_X(t)$ and evaluating it at $t = 0$.

$$M_X'(t) = \frac{pe^t}{(1-(1-p)e^t)^2}$$

$$M_X'(0) = \frac{p}{p^2} = \frac{1}{p} = \mathbf{E}(X)$$