Stat 2605 Tutorial 7

November 22, 2022

1. Suppose X and Y are independent with:

$$\mathbf{E}(X) = 2, \quad \mathbf{E}(Y) = 3, \quad \mathbf{Var}(X) = 1, \quad \mathbf{Var}(Y) = 2$$

Let $Z = 2X - Y + 1$. Use Chebyshev's inequality to find a lower bound for $\mathbf{P}(|Z - 2| < 5)$.

First, note that

$$\begin{aligned} \mathbf{E}(Z) &= 2\mathbf{E}(X) - \mathbf{E}(Y) + 1 = 2. \\ \mathbf{Var}(Z) &= 4\mathbf{Var}(X) + \mathbf{Var}(Y) = 6 \end{aligned}$$

Chebyshev's inequality states that

$$\mathbf{P}(|X - \mu_X| < k) \geq 1 - \frac{\sigma_X^2}{k^2}$$

In our case, $\mu_Z = 2$, $k = 5$, and $\sigma_Z^2 = 6$. Therefore,

$$\mathbf{P}(|Z - 2| < 5) \geq 1 - \frac{6}{5^2} = \frac{19}{25}$$

2. Suppose check-ins at a small hotel occur satisfying the three conditions of a Poisson process, $N(t)$. The rate is $\lambda = 6$ and the unit of time is an hour. Find the probability that there are more than 2 check-ins within 15 minutes.

15 minutes is 0.25 hours. Thus $\lambda t = 6 \cdot 0.25 = 1.5$. Then the pmf of $N(0.25)$ is given by

$$f(x) = \frac{(\lambda t)^x e^{-\lambda t}}{x!} = \frac{1.5^x e^{-1.5}}{x!},$$

for $x = 0, 1, 2, \ldots$.

$$\begin{aligned} \mathbf{P}(N(0.25) \geq 2) &= 1 - \mathbf{P}(N(0.25) < 2) \\ &= 1 - \mathbf{P}(N(0.25) = 0) - \mathbf{P}(N(0.25) = 1) \\ &= 1 - \frac{1.5^0 e^{-1.5}}{0!} - \frac{1.5^1 e^{-1.5}}{1!} \\ &= 0.44217 \end{aligned}$$

3. Continuing from the setting of the previous problem, let T_2 and T_3 be inter-arrival times, where T_2 is the time between first and second arrivals, and T_3 is the time between second and third arrivals. Determine the joint pdf of T_2 and T_3. Find $\mathbf{Var}\,(T_2 + T_3)$.

   We know that all inter-arrival times are i.i.d. exponentially distributed random variables with rate parameter equal to λ. The marginal pdfs for T_2 and T_3 are given, respectively, as:

$$f(t_2) = \begin{cases} \lambda e^{-\lambda t_2} & t_2 > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad f(t_3) = \begin{cases} \lambda e^{-\lambda t_3} & t_3 > 0 \\ 0 & \text{otherwise} \end{cases}$$

   As T_2 and T_3 are independent, their joint distribution is simply the product of their marginals.

$$f(t_2,\, t_3) \;=\; f(t_2) \cdot f(t_3) \;=\; \begin{cases} \lambda^2 e^{-\lambda(t_2+t_3)} & t_2 > 0, \quad t_3 > 0 \\ 0 & \text{otherwise} \end{cases}$$

   Finally, as T_2 and T_3 are independent random variables, the variance of their sum is the sum of their variances.

$$\mathbf{Var}\,(T_2 + T_3) \;=\; \mathbf{Var}\,(T_2) \;+\; \mathbf{Var}\,(T_3) \;=\; \frac{1}{\lambda^2} \;+\; \frac{1}{\lambda^2} \;=\; \frac{2}{\lambda^2}$$

4. Suppose X and Y are independent exponential random variables with:

$$\mathbf{E}\,(X) = 2, \quad \mathbf{E}\,(Y) = 3$$

   (a) Find $\mathbf{E}\,(XY)$.

   Since X and Y are independent random variables,

$$\mathbf{E}\,(XY) \;=\; \mathbf{E}\,(X) \cdot \mathbf{E}\,(Y) \;=\; 2 \cdot 3 \;=\; 6$$

   (b) Find $\mathbf{E}\left(X^2 Y^2\right)$.

   Using the fact that X and Y are independent random variables, and that the variance of an exponential random variable is the square of its mean,

$$\begin{aligned}
\mathbf{E}\left(X^2Y^2\right) &= \mathbf{E}\left(X^2\right)\mathbf{E}\left(Y^2\right) \\
&= \left(\mathbf{Var}\,(X) + (\mathbf{E}\,(X))^2\right)\left(\mathbf{Var}\,(Y) + (\mathbf{E}\,(Y))^2\right) \\
&= \left((\mathbf{E}\,(X))^2 + (\mathbf{E}\,(X))^2\right)\left((\mathbf{E}\,(Y))^2 + (\mathbf{E}\,(Y))^2\right) \\
&= (2(\mathbf{E}\,(X))^2)(2(\mathbf{E}\,(Y))^2) \\
&= (2 \cdot 2^2)(2 \cdot 3^2) \\
&= 144
\end{aligned}$$

(c) Find $\mathbf{E}\left(X^2Y\right)$.

We employ a combination of techniques seen in (a) and (b).

$$\begin{aligned}
\mathbf{E}\left(X^2Y\right) &= \mathbf{E}\left(X^2\right)\mathbf{E}\left(Y\right) \\
&= \left(\mathbf{Var}\left(X\right) + (\mathbf{E}\left(X\right))^2\right)\mathbf{E}\left(Y\right) \\
&= \left((\mathbf{E}\left(X\right))^2 + (\mathbf{E}\left(X\right))^2\right)\mathbf{E}\left(Y\right) \\
&= (2(\mathbf{E}\left(X\right))^2)\mathbf{E}\left(Y\right) \\
&= (2 \cdot 2^2)(3) \\
&= 24
\end{aligned}$$