Stat 2605 Tutorial 8

December 6, 2022

Note: for readability, I use X_t rather than $X(t)$.

1. New students come to a registration center as a Poisson process N_t. On average, 40 students arrive in 15 minutes. Find the probability that there are at least four students arriving within two minutes.

2. Suppose $\{X_t : t = 0, 1, 2, 3, \ldots\}$ is a time homogeneous Markov chain with the set of states $S = \{1, 2\}$ and one step transition probability matrix:

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix},$$

where p_{ij} is the probability of transitioning from state i to state j, which we may also denote using $\mathbf{P}(i \to j)$.

   (a) Suppose $\mathbf{P}(X_0 = 2) = 1$. Find

$$\mathbf{P}(X_3 = 2,\, X_2 = 1,\, X_1 = 1,\, X_0 = 2).$$

   (b) Suppose $\mathbf{P}(X_0 = 2) = 1/3$. Find

$$\mathbf{P}(X_3 = 1,\, X_2 = 2,\, X_1 = 1,\, X_0 = 2).$$

   (c) Find the probability distribution π_2 of X_2 if X_0 has probability distribution

$$\pi_0 = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$$

Property of the Markov chain (will not be proven): Suppose that X_t is a Markov chain for time $t = 0, 1, 2, \ldots$. Then for any $t_1 < t_2 < t_3 < \ldots$ (where $t_1, t_2, t_3, \ldots$ are not necessarily consecutive integers), we have:

$$\mathbf{P}\left(X_{t_{n+1}} = i_{n+1} \,|\, X_{t_n} = i_n,\, X_{t_{n-1}} = i_{n-1},\, \ldots,\, X_{t_1} = i_1\right) = \mathbf{P}\left(X_{t_{n+1}} = i_{n+1} \,|\, X_{t_n} = i_n\right)$$

In other words, the probability of a state at a future time step given the states of the past time steps, depends only on the state at the most recent conditioned time step.

Example: Consider a two-step transition probability with state space $S = \{1, 2, 3, 4\}$. From the results of the previous property:

$$\mathbf{P}(X_4 = 2 \,|\, X_2 = 3,\, X_0 = 1) = \mathbf{P}(X_4 = 2 \,|\, X_2 = 3)$$

3. Continuing with $\{X_t\}$ as defined in Question 2, let $\mathbf{P}\left(X_0 = 2)\right) = 1/2$. Find

$$\mathbf{P}\left(X_5 = 1,\, X_2 = 2,\, X_0 = 2\right).$$

4. Consider again, the Markov chain introduced in Question 2.
   (a) Is this Markov chain ergodic?
   (b) Find its steady-state probabilities.