Stat 2605 Tutorial 8

December 6, 2022

1. New students come to a registration center as a Poisson process N_t. On average, 40 students arrive in 15 minutes. Find the probability that there are at least four students arriving within two minutes.

   Consider one minute as the unit of time. Then

   $$\mu = \lambda t = \frac{40}{15} \times 2 = \frac{16}{3}.$$

   As such,

   $$N_2 \sim \text{Poisson}(\mu = 16/3)$$

   $$\begin{aligned}
   \mathbf{P}(N_2 \geq 4) &= 1 - \mathbf{P}(N_2 < 4) \\
   &= 1 - \mathbf{P}(N_2 \leq 3) \\
   &= 1 - (\mathbf{P}(N_2 = 0) + \mathbf{P}(N_2 = 1) + \mathbf{P}(N_2 = 2) + \mathbf{P}(N_2 = 3)) \\
   &= 1 - e^{-16/3}\left(\frac{(16/3)^0}{0!} + \frac{(16/3)^1}{1!} + \frac{(16/3)^2}{2!} + \frac{(16/3)^3}{3!}\right) \\
   &= 1 - 0.2213 \\
   &= 0.7787
   \end{aligned}$$

2. Suppose $\{X_t : t = 0, 1, 2, 3, \ldots\}$ is a time homogeneous Markov chain with the set of states $S = \{1, 2\}$ and one step transition probability matrix:

   $$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix},$$

   where p_{ij} is the probability of transitioning from state i to state j, which we may also denote using $\mathbf{P}(i \to j)$.

   (a) Suppose $\mathbf{P}(X_0 = 2) = 1$. Find

   $$\mathbf{P}(X_3 = 2, X_2 = 1, X_1 = 1, X_0 = 2).$$

The sequence we need to consider is:

$$\underset{0}{2} \rightarrow \underset{1}{1} \rightarrow \underset{2}{1} \rightarrow \underset{3}{2}$$

$$\begin{aligned} &\mathbf{P}\left(X_3 = 2,\, X_2 = 1,\, X_1 = 1,\, X_0 = 2\right) \\ =\ &\mathbf{P}\left(X_0 = 2\right) \cdot \mathbf{P}\left(2 \to 1\right) \cdot \mathbf{P}\left(1 \to 1\right) \cdot \mathbf{P}\left(1 \to 2\right) \\ =\ &1 \cdot p_{21} \cdot p_{11} \cdot p_{12} \\ =\ &1 \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{2}{3} \\ =\ &\frac{2}{36} = \frac{1}{18} \end{aligned}$$

(b) Suppose $\mathbf{P}\left(X_0 = 2\right) = 1/3$. Find

$$\mathbf{P}\left(X_3 = 1,\, X_2 = 2,\, X_1 = 1,\, X_0 = 2\right).$$

The sequence we need to consider is:

$$\underset{0}{2} \rightarrow \underset{1}{1} \rightarrow \underset{2}{2} \rightarrow \underset{3}{1}$$

$$\begin{aligned} &\mathbf{P}\left(X_3 = 1,\, X_2 = 2,\, X_1 = 1,\, X_0 = 2\right) \\ =\ &\mathbf{P}\left(X_0 = 2\right) \cdot \mathbf{P}\left(2 \to 1\right) \cdot \mathbf{P}\left(1 \to 2\right) \cdot \mathbf{P}\left(2 \to 1\right) \\ =\ &\frac{1}{3} \cdot p_{21} \cdot p_{12} \cdot p_{21} \\ =\ &\frac{1}{3} \cdot \frac{1}{4} \cdot \frac{2}{3} \cdot \frac{1}{4} \\ =\ &\frac{2}{144} = \frac{1}{72} \end{aligned}$$

(c) Find the probability distribution π_2 of X_2 if X_0 has probability distribution

$$\pi_0 = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$$

The probability distribution for π_2 is computed as $\pi_0 \cdot P^2$.

$$\begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} 79/288 & 209/288 \end{bmatrix}$$

Property of the Markov chain (will not be proven): Suppose that X_t is a Markov chain for time $t = 0, 1, 2, \ldots$. Then for any $t_1 < t_2 < t_3 < \ldots$ (where $t_1, t_2, t_3, \ldots$ are not necessarily consecutive integers), we have:

$$\mathbf{P}\left(X_{t_{n+1}} = i_{n+1} \,|\, X_{t_n} = i_n,\, X_{t_{n-1}} = i_{n-1},\, \ldots,\, X_{t_1} = i_1\right) = \mathbf{P}\left(X_{t_{n+1}} = i_{n+1} \,|\, X_{t_n} = i_n\right)$$

In other words, the probability of a state at a future time step given the states of the past time steps, depends only on the state at the most recent conditioned time step.

Example: Consider a two-step transition probability with state space $S = \{1, 2, 3, 4\}$. From the results of the previous property:

$$\mathbf{P}(X_4 = 2 \mid X_2 = 3,\ X_0 = 1) \ = \ \mathbf{P}(X_4 = 2 \mid X_2 = 3)$$

3. Continuing with $\{X_t\}$ as defined in Question 2, let $\mathbf{P}(X_0 = 2)) = 1/2$. Find

$$\mathbf{P}(X_5 = 1,\ X_2 = 2,\ X_0 = 2)\,.$$

We begin by determining the two-step and three-step transition probabilities.

$$P^{(2)} \ = \ P^2 \ = \ \begin{bmatrix} p_{11}^{(2)} & p_{12}^{(2)} \\ p_{21}^{(2)} & p_{22}^{(2)} \end{bmatrix} \ = \ \begin{bmatrix} 0.2778 & 0.7222 \\ 0.2708 & 0.7292 \end{bmatrix}$$

$$P^{(3)} \ = \ P^3 \ = \ \begin{bmatrix} p_{11}^{(3)} & p_{12}^{(3)} \\ p_{21}^{(3)} & p_{22}^{(3)} \end{bmatrix} \ = \ \begin{bmatrix} 0.2731 & 0.7269 \\ 0.2726 & 0.7274 \end{bmatrix}$$

We cannot perform a procedure similar to that performed in (2a) and (2b) because we do not know the states at time steps 1, 3, and 4. However, we can decompose the probability we seek into conditional probabilities and apply the Markov chain property mentioned above.

$$\begin{aligned} &\mathbf{P}(X_5 = 1,\ X_2 = 2,\ X_0 = 2) \\ =\ &\mathbf{P}(X_5 = 1 \mid X_2 = 2,\ X_0 = 2) \cdot \mathbf{P}(X_2 = 2,\ X_0 = 2) \\ =\ &\mathbf{P}(X_5 = 1 \mid X_2 = 2) \cdot \mathbf{P}(X_2 = 2,\ X_0 = 2) \\ =\ &\mathbf{P}(X_5 = 1 \mid X_2 = 2) \cdot \mathbf{P}(X_2 = 2 \mid X_0 = 2) \cdot \mathbf{P}(X_0 = 2) \\ =\ &p_{21}^{(3)} \cdot p_{22}^{(2)} \cdot \frac{1}{2} \\ =\ &0.2726 \times 0.7292 \times \frac{1}{2} \\ =\ &0.0994 \end{aligned}$$

4. Consider again, the Markov chain introduced in Question 2.

   (a) Is this Markov chain ergodic?

   A Markov chain is ergodic if the k^{th}-step transition matrix has all positive entries for some $k \geq 1$. Clearly, taking $k = 1$, P has all positive entries. Therefore, this Markov chain is ergodic.

(b) Find its steady-state probabilities.

As $\{X_t\}$ is an ergodic Markov chain, as $t \to \infty$, the distribution π_t of X_t will converge to

$$\pi = \begin{bmatrix} a & b \end{bmatrix}.$$

We can find the steady state probabilities (π) by solving the system of equations:

$$\begin{cases} \pi P & = \pi \\ \sum_{i=1}^{n} \pi_i & = 1 \end{cases}$$

Similar to the example presented in lecture 23:

- We have 3 equations and 2 unknowns.
- We can get rid of one redundant equation, from $\pi P = \pi$.

From $\pi P = \pi$, we have:

$$\frac{1}{3}a + \frac{1}{4}b = a \tag{1}$$

$$\frac{2}{3}a + \frac{3}{4}b = b \tag{2}$$

Additionally, we have that:

$$a + b = 1 \tag{3}$$

We will ignore the redundant equation (2) and use only (1) and (3).

Rearranging (3), we obtain $b = 1 - a$, which we can then substitute into (1).

$$\begin{aligned} & \frac{1}{3}a + \frac{1}{4}(1-a) && = a \\ \iff & \frac{1}{3}a + \frac{1}{4} - \frac{1}{4}a && = a \\ \iff & a + \frac{1}{4}a - \frac{1}{3}a && = \frac{1}{4} \\ \iff & \left(\frac{12}{12} + \frac{3}{12} - \frac{4}{12}\right)a && = \frac{1}{4} \\ \iff & \frac{11}{12}a && = \frac{1}{4} \\ \iff & a && = \frac{3}{11} \end{aligned}$$

Substituting $a = 3/11$ into (3), we obtain that $b = 8/11$. Therefore, the steady state probabilities are $\pi = \begin{bmatrix} 3/11 & 8/11 \end{bmatrix}$.