Tutorial 9: Solutions

March 21, 2018

Question 9.1.6, Page 372

An experiment to compare the tension bond strength of polymer latex modified mortar (Portland cement mortar to which polymer latex emulsions have been added during mixing) to that of unmodified mortar resulted in $\overline{x} = 18.12$ kgf/cm^2 for the modified mortar ($m = 40$) and $\overline{y} = 16.87$ kgf/cm^2 for the unmodified mortar ($n = 32$). Let μ_1 and μ_2 be the true average tension bond strengths for the modified and unmodified mortars, respectively. Assume that the bond strength distributions are both normal.

(a) Assuming that $\sigma_1 = 1.6$ and $\sigma_2 = 1.4$, test $H_0 : \mu_1 - \mu_2 \leq 0$ versus $H_A : \mu_1 - \mu_2 > 0$ at level $\alpha = 0.01$.

Since both of our samples are normally distributed with known population standard deviations, we can proceed to find a test statistic that is exactly standard normal:

$$z = \frac{\overline{x} - \overline{y} - \Delta_0}{\sqrt{\frac{{\sigma_1}^2}{m} + \frac{{\sigma_2}^2}{n}}} = \frac{18.12 - 16.87 - 0}{\sqrt{\frac{1.6^2}{40} + \frac{1.4^2}{32}}} = 3.532$$

This is an upper-tailed test. We require $z_{1-\alpha} = z_{0.99} = 2.3263$. We reject H_0 if $z > z_{1-\alpha}$. Since $3.532 > 2.3263$, we reject H_0 . We conclude at the 1% level of significance that the true mean strength of modified mortar is greater than that of unmodified mortar.

(b) Compute the probability of a type II error for the test of part (a) when $\mu_1 - \mu_2 = 1$.

Let $\sigma_d = \sqrt{\frac{{\sigma_1}^2}{m} + \frac{{\sigma_2}^2}{n}}$, and let $\Delta' = \mu_1 - \mu_2 = 1$. We keep α as 0.01. Then:

$$\begin{aligned}
\beta(1) &= \mathbf{P}\left(\text{Type II Error} \mid \Delta' = 1\right) \\
&= \mathbf{P}\left(\text{Fail to reject } H_0 \text{ when } H_0 \text{ false} \mid \Delta' = 1\right) \\
&= \mathbf{P}\left(\frac{\overline{X} - \overline{Y} - \Delta_0}{\sigma_d} < z_{1-\alpha} \middle| \Delta' = 1\right) \\
&= \mathbf{P}\left(\overline{X} - \overline{Y} < z_{1-\alpha}\sigma_d + \Delta_0 \middle| \Delta' = 1\right) \\
&= \mathbf{P}\left(\frac{\overline{X} - \overline{Y} - \Delta'}{\sigma_d} < \frac{z_{1-\alpha}\sigma_d + \Delta_0 - \Delta'}{\sigma_d}\right) \\
&= \mathbf{P}\left(Z < z_{1-\alpha} + \frac{\Delta_0 - \Delta'}{\sigma_d}\right)
\end{aligned}$$

*We can factor out a -1 in the numerator of the fraction to match the form given in the textbook, but my formulation is equivalent to theirs.

$$= \mathbf{P}\left(Z < 2.3263 + \frac{0 - 1}{\sqrt{\frac{1.6^2}{40} + \frac{1.4^2}{32}}}\right)$$

$$= \Phi(-0.4993)$$
$$= 0.30878$$

(c) Suppose the investigator decided to use a level 0.05 test and wished $\beta = 0.10$ when $\mu_1 - \mu_2 = 1$. If $m = 40$, what value of n is necessary?

From the formulation derived in (b), it is clear that $\beta(1) = 0.10$ implies that:

$$\Phi\left(z_{1-\alpha} + \frac{\Delta_0 - \Delta'}{\sqrt{\frac{{\sigma_1}^2}{m} + \frac{{\sigma_2}^2}{n}}}\right) = 0.10$$

$$z_{1-\alpha} + \frac{\Delta_0 - \Delta'}{\sqrt{\frac{{\sigma_1}^2}{m} + \frac{{\sigma_2}^2}{n}}} = -1.2815$$

$$\vdots \quad \text{Lots of rearranging}$$

$$n = \frac{{\sigma_2}^2}{\left(\frac{\Delta_0 - \Delta'}{-1.2815 - z_{1-\alpha}}\right)^2 - \frac{{\sigma_1}^2}{m}}$$

$$= \frac{1.4^2}{\left(\frac{0 - 1}{-1.2815 - 1.6449}\right)^2 - \frac{1.6^2}{40}}$$

$$= 37.142 \to 38$$

(d) How would the analysis and conclusion of part (a) change if σ_1 and σ_2 were unknown but $s_1 = 1.6$ and $s_2 = 1.4$?

If we use s_1 and s_2 instead of σ_1 and σ_2, in order to use the large sample test statistic, we require that **both** m and n are greater than 40. Since we only have that m is greater than 40, we cannot proceed using the large sample test statistic. However, since it was given that the underlying population of both samples is normal, we can use the t-test statistic introduced in section 9.2.
The values of s_1 and s_2 are identical in magnitude with σ_1 and σ_2, respectively. Therefore our new test statistic is unchanged and we obtain that $t = 3.532$, as in (a). We calculate our degrees of freedom to be:

$$\nu = \frac{\left(\frac{{s_1}^2}{m} + \frac{{s_2}^2}{n}\right)^2}{\frac{({s_1}^2/m)^2}{m-1} + \frac{({s_2}^2/n)^2}{n-1}} = \frac{\left(\frac{1.6^2}{40} + \frac{1.4^2}{32}\right)^2}{\frac{(1.6^2/40)^2}{39} + \frac{(1.4^2/32)^2}{31}}$$

$$= 69.4005 \to 69 \quad \text{(must round down to nearest integer if using textbook chart)}$$

As usual, since this is an upper-tailed test, we require $t_{\nu,\alpha} = t_{69,0.01}$. $t_{60,0.01} = 2.390$ and we can see that $t_{69,0.01}$ will be some value slightly less than 2.390. We reject H_0 if $t > t_{69,0.01}$. Since $3.532 > 2.390$, we can once again reject H_0 at the 1% level of significance.

Question 9.1.14, Page 373

The level of monoamine oxidase (MAO) activity in blood platelets (nm/mg protein/h) was determined for each individual in a sample of 43 chronic schizophrenics, resulting in $\overline{x} = 2.69$ and $s_1 = 2.30$, as well as for 45 normal subjects, resulting in $\overline{y} = 6.35$ and $s_2 = 4.03$. Does this data strongly suggest that true average MAO activity for normal subjects is more than twice the activity level for schizophrenics? Derive a test procedure and carry out the test using $\alpha = 0.01$. [Hint: H_0 and H_A here have a different form from the three standard cases. Let μ_1 and μ_2 refer to true average MAO activity for schizophrenics and normal subjects, respectively, and consider the parameter $\theta = 2\mu_1 - \mu_2$. Write H_0 and H_A in terms of θ, estimate θ, and derive $\sigma_{\hat{\theta}}$.

We are told that we want to test if "normal activity is more than twice of schizophrenic". If we define $\theta = 2\mu_1 - \mu_2$ then we obtain the hypotheses $H_0 : \theta \geq 0$, $H_A : \theta < 0$.

We proceed to estimate θ using $\widehat{\theta} = 2\overline{X} - \overline{Y}$ (since $\mathbf{E}\left(\widehat{\theta}\right) = \theta$). Then

$$\mathbf{Var}\left(\widehat{\theta}\right) = \mathbf{Var}\left(2\overline{X} - \overline{Y}\right) = 4\mathbf{Var}\left(\overline{X}\right) + \mathbf{Var}\left(\overline{Y}\right) = \frac{4{\sigma_1}^2}{m} + \frac{{\sigma_2}^2}{n}$$

Suppose that both samples were normally distributed. Then we would have the test statistic

$$Z = \frac{\widehat{\theta} - \theta_0}{\sigma_{\widehat{\theta}}} = \frac{(2\overline{X} - \overline{Y}) - \theta_0}{\sqrt{\frac{4{\sigma_1}^2}{m} + \frac{{\sigma_2}^2}{n}}}$$

It is not known whether our two samples are normally distributed nor do know either population standard deviations. However, since we have $m,\ n > 40$, we can approximate the population standard deviations using the sample standard deviations. In addition, Central Limit Theorem tells us that $\overline{X}$ and $\overline{Y}$ approach an approximate normal distribution. As such, $\widehat{\theta} = 2\overline{X} - \overline{Y}$, which is a linear combination of approximately normal random variables, will also approach an approximate normal distribution for m and n both sufficiently large. Thus our new test statistic is:

$$Z = \frac{2\overline{X} - \overline{Y} - \theta_0}{\sqrt{\frac{4{s_1}^2}{m} + \frac{{s_2}^2}{n}}} \quad \overset{\text{approx.}}{\sim} N(0,1)$$

Plugging in the data, we obtain:

$$z = \frac{2(2.69) - 6.35 - 0}{\sqrt{\frac{4(2.30)^2}{43} + \frac{(4.03)^2}{45}}} = -1.050$$

This is a lower-tailed test so we require $-z_{1-\alpha} = -z_{0.99} = -2.263$. We reject H_0 if $z < -z_{1-\alpha}$. Since $-1.050 \nless -2.3263$, we fail to reject H_0 . We conclude at the 1% significance level that there is insufficient evidence that the true average MAO activity for normal subjects is more than twice the true average MAO activity for schizophrenics.

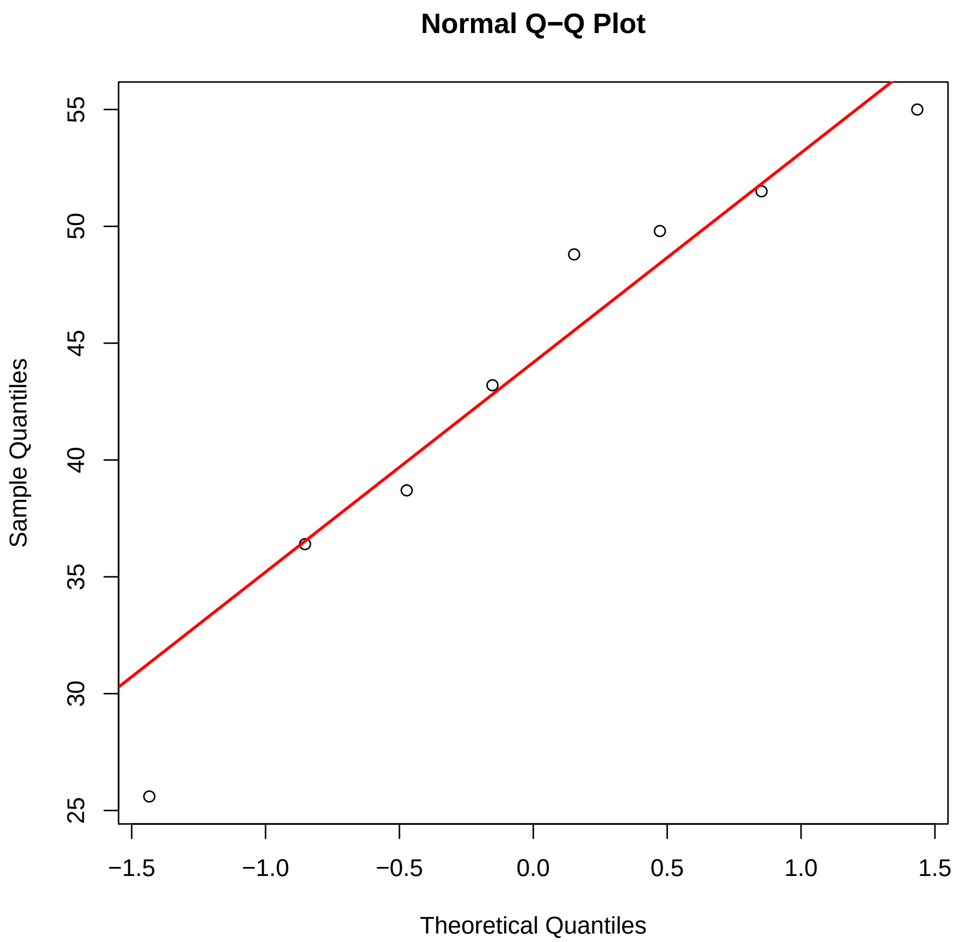
Question 9.3.36, Page 388

Consider the accompanying data on breaking load (kg/25 mm width) for various fabrics in both an unabraded condition and an abraded condition. Use the paired t-test to test $H_0 : \mu_D \leq 0$ versus $H_A : \mu_D > 0$ at significance level $\alpha = 0.01$.

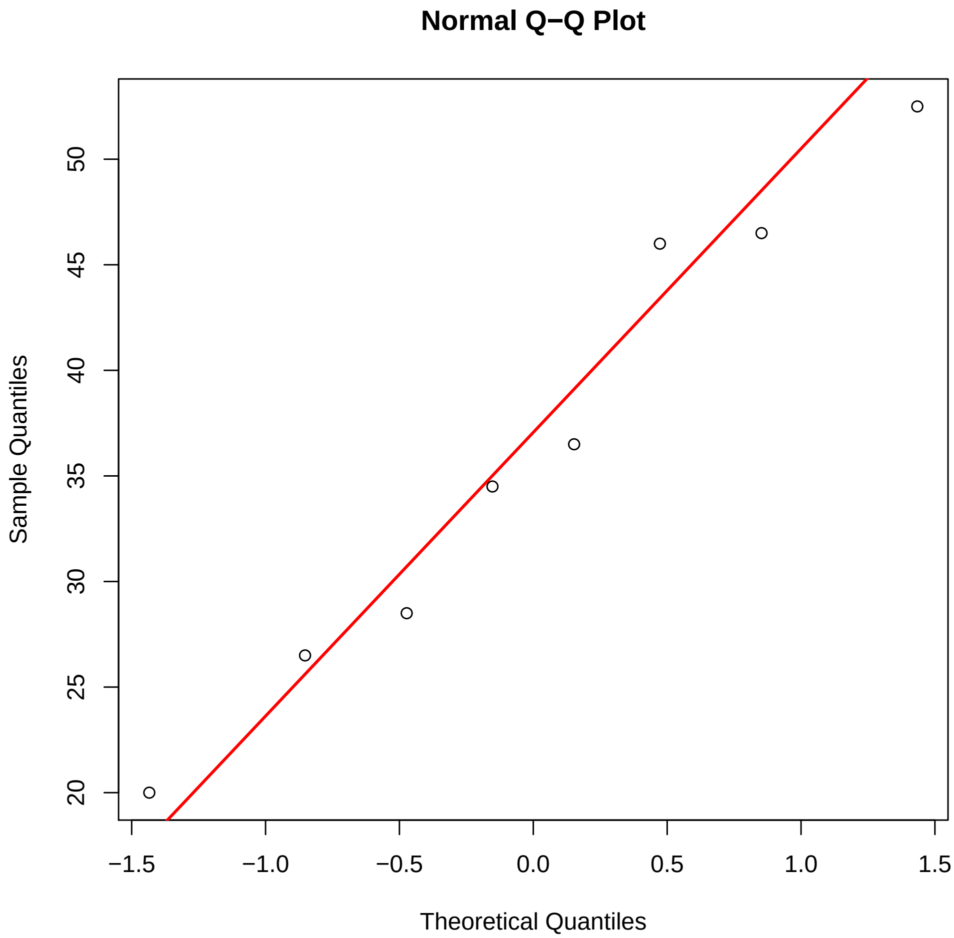
U	36.4	55.0	51.5	38.7	43.2	48.8	25.6	49.8
A	28.5	20.0	46.0	34.5	36.5	52.5	26.5	46.5

We first create QQ plots for the U and A datasets.

QQ Plot for Unabraded Fabrics



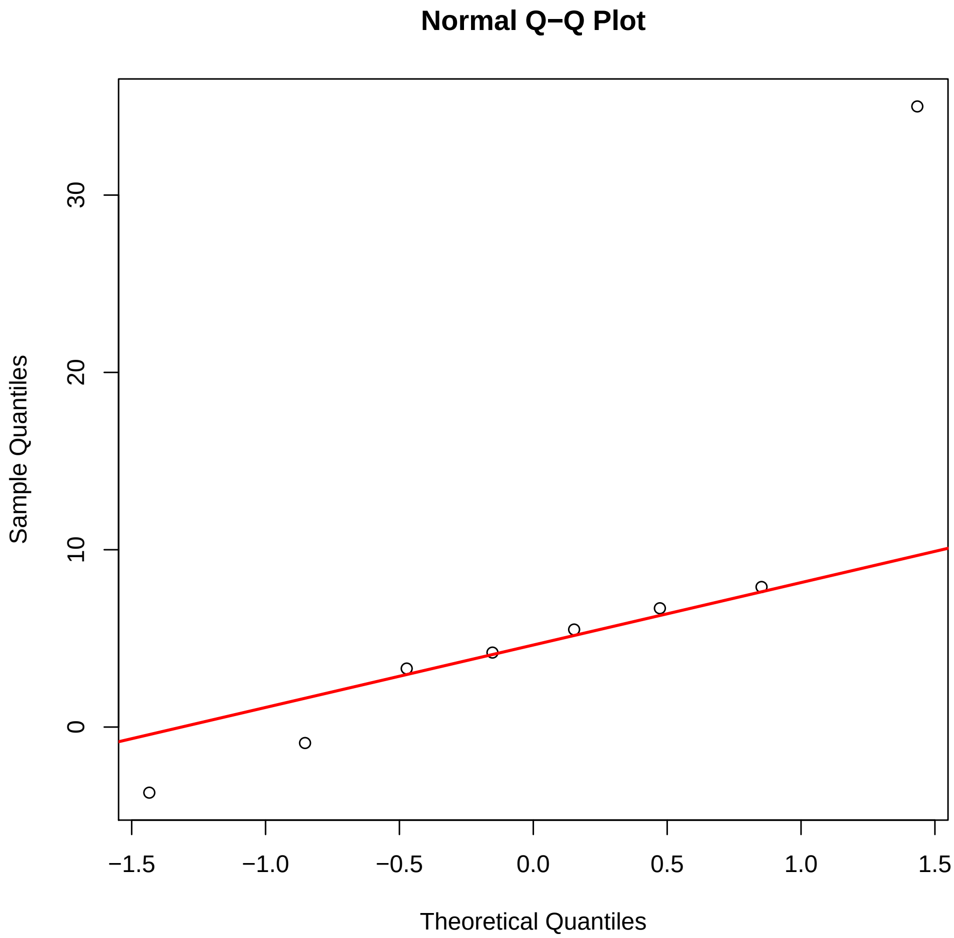
QQ Plot for Abraded Fabrics



It seems that it is a reasonable assumption to say that the datasets U and A are have an underlying normal distribution. Since D = U - A is a linear combination of normal random variables, theoretically, D should also have an underlying normal distribution. We check this by first producing the dataset D and then making the QQ plot.

D	7.9	35.0	5.5	4.2	6.7	-3.7	-0.9	3.3

QQ Plot for D = U - A Fabrics



We observe that most of our points lie quite close to the line and we have one very large outlier. We will ignore this outlier and proceed to assume that D has an underlying normal distribution. The hypotheses that we are interested in testing are: $H_0 : \mu_D \leq 0$, $H_A : \mu_D > 0$. Using R, we calculate $\overline{d} = 7.25$ and $s_D = 11.863....$

$$t = \frac{\overline{d} - \Delta_0}{s_D/\sqrt{n}} = \frac{7.25 - 0}{11.863.../\sqrt{8}} = 1.7286$$

As this is an upper-tailed test, we need $t_{n-1,\alpha} = t_{7,0.01} = 2.998$. We reject H_0 if $t > t_{7,0.01}$. Since $1.7286 \not> 2.998$, we fail to reject H_0 . We conclude at the 1% level of significance that there is insufficient evidence that unabraded breaking load is greater than abraded breaking load.