Tutorial 1

January 23, 2020

Question 1

Suppose that vehicles taking a particular highway exit can turn right (R), left (L), or go straight (S), with equal probabilities. Consider observing the direction of each of three successive vehicles.

(a) How many outcomes are possible? Do not list them! List the outcomes in the event A that all vehicles go in the same direction. Compute $\mathbf{P}(A)$.

Let us express the turns of the three cars as a tuple. Let T represent the total collection of possible outcomes for three exiting cars:

$$T = \big\{(t_1, t_2, t_3) : t_1, t_2, t_3 \in \{\mathrm{L}, \mathrm{S}, \mathrm{R}\}\big\}$$

The first car has three outcomes. For each outcome of the first car, the second car has three outcomes. For each outcome of the first and second cars, the third car has three outcomes. The total number of possible outcomes for the three exiting cars is:

$$n(T) = 3 \cdot 3 \cdot 3 = 27$$

Let A be the event that all vehicles go the same direction:

$$A = \big\{(\mathrm{L}, \mathrm{L}, \mathrm{L}), (\mathrm{S}, \mathrm{S}, \mathrm{S}), (\mathrm{R}, \mathrm{R}, \mathrm{R})\big\}$$

The number of outcomes in A is $n(A) = 3$. Then the probability that all vehicles go the same direction is:

$$\mathbf{P}(A) = \frac{n(A)}{n(T)} = \frac{3}{27} = \frac{1}{9}$$

(b) List all the outcomes in the event B that all three vehicles take different directions. Compute $\mathbf{P}(B)$.

$$B = \big\{(\mathrm{L}, \mathrm{S}, \mathrm{R}), (\mathrm{L}, \mathrm{R}, \mathrm{S}), (\mathrm{S}, \mathrm{L}, \mathrm{R}), (\mathrm{S}, \mathrm{R}, \mathrm{L}), (\mathrm{R}, \mathrm{S}, \mathrm{L}), (\mathrm{R}, \mathrm{L}, \mathrm{S})\big\}$$

The number of outcomes in B is $n(B) = 6$. Then the probability that all vehicles go in different directions is:

$$\mathbf{P}(B) = \frac{n(B)}{n(T)} = \frac{6}{27} = \frac{2}{9}$$

Question 2

Assume that 55% of the adult population consumes coffee, 45% tea, and 70% at least one of the two drinks.

(a) What is the probability that a randomly selected adult consumes both coffee and tea?

Let C represent the set of coffee drinking people and let T represent the set of tea drinking people. From the question, we are given that:

- $\mathbf{P}(C) = 0.55$
- $\mathbf{P}(T) = 0.45$
- $\mathbf{P}(C \cup T) = 0.70$

We want to find the probability that a randomly selected person drinks both coffee and tea, $\mathbf{P}(C \cap T)$. Recall that for any two events A and B, the inclusion-exclusion principle states:

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$$

Rearranging, we obtain:

$$\mathbf{P}(A \cap B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cup B)$$

Applying this to the given context, we have:

$$\begin{aligned}\mathbf{P}(C \cap T) &= \mathbf{P}(C) + \mathbf{P}(T) - \mathbf{P}(C \cup T) \\ &= 0.55 + 0.45 - 0.70 \\ &= 0.30\end{aligned}$$

The probability that a randomly selected person drinks both coffee and tea is 0.30.

(b) What is the probability that he or she consumes only one drink?

To find the set of people who drink only coffee or only tea, we look for the set of people who *drink coffee and not tea*, and the set of people who *drink tea and not coffee*.

Through the use of venn diagrams, it can easily be shown that:

$$C \cup T = (C \cap T^c) \cup (T \cap C^c) \cup (C \cap T)$$

We have now expressed $C \cup T$ as a union of disjoint sets. By the axioms of probability, the probability of a union of disjoint sets is the sum of the individual probabilities:

$$\mathbf{P}(C \cup T) = \mathbf{P}(C \cap T^c) + \mathbf{P}(T \cap C^c) + \mathbf{P}(C \cap T)$$

We notice that the first two terms of the right hand side are exactly what we were looking for. Rearranging, we obtain:

$$\begin{aligned}\mathbf{P}(C \cap T^c) + \mathbf{P}(T \cap C^c) &= \mathbf{P}(C \cup T) - \mathbf{P}(C \cap T) \\ &= 0.70 - 0.30 \\ &= 0.40\end{aligned}$$

The probability that a randomly selected person drinks only one of coffee or tea is 0.40.

Question 3

A production facility employs 20 workers on day shift, 15 on swing shift, and 10 on graveyard shift. A quality control consultant is using the database to randomly select 6 of these workers for in-depth interviews.

(a) How many selections result in all the 6 workers coming from day shift?

Let A represent the event that all six people selected are from the day shift. We have 20 day shift workers to choose from. This is a combination as order does not matter - interviewing Alice then Bob will produce the same result as interviewing Bob then Alice.

$$n(A) = \binom{20}{6} = 38760$$

(b) What is the probability that all the 6 selected workers will be from the same shift?

Let B represent the collection of six-member groups where each member is from the same shift. There are three cases to consider in order to calculate $n(B)$: all six members come from either the day-shift, swing shift, or graveyard shift.

$$n(B) = \binom{20}{6} + \binom{15}{6} + \binom{10}{6} = 43975$$

Let T represent the collection of all possible six-member groups.

$$n(T) = \binom{45}{6} = 8145060$$

$$\mathbf{P}(B) = \frac{n(B)}{n(T)} = \frac{43975}{8145060} \approx 0.00540$$

Question 4

An automobile service facility specializing in engine tune-ups knows that 45% of all tune ups are done on 4-cylinder cars, 40% on 6-cylinder cars, and 15% on 8-cylinder cars. Let X be the number of cylinders on the next car to be tuned up.

(a) Find the pdf and cdf of X. Draw the pdf and cdf of X.

From the information given in the question, X is a discrete random variable. Therefore its probability mass function represents the probability of X taking on some value x. The pmf of X is:

$$f(x) = \mathbf{P}(X = x) = \begin{cases} 0.45 & x = 4 \\ 0.40 & x = 6 \\ 0.15 & x = 8 \\ 0 & \text{otherwise} \end{cases}$$

$$f : \mathbb{R} \to [0, 1]$$

The cumulative mass function is the cumulative sum of the probabilities of $f(x)$:

$$F(x) = \mathbf{P}(X \leq x) = \begin{cases} 0 & x < 4 \\ 0.45 & 4 \leq x < 6 \\ 0.85 & 6 \leq x < 8 \\ 1 & x \geq 8 \end{cases}$$

$$F : \mathbb{R} \to [0, 1]$$

See last page for graphs.

(b) What is the probability that the next car has:

(i) at least 6 cylinders?

Method 1: Using pmf

$$\begin{aligned} \mathbf{P}(X \geq 6) &= \mathbf{P}(X = 6) + \mathbf{P}(X = 7) + \mathbf{P}(X = 8) \\ &= 0.40 + 0 + 0.15 \\ &= 0.55 \end{aligned}$$

Method 2: Using cmf

$$\begin{aligned} \mathbf{P}(X \geq 6) &= 1 - \mathbf{P}(X < 6) \\ &= 1 - \mathbf{P}(X \leq 5) \\ &= 1 - 0.45 \\ &= 0.55 \end{aligned}$$

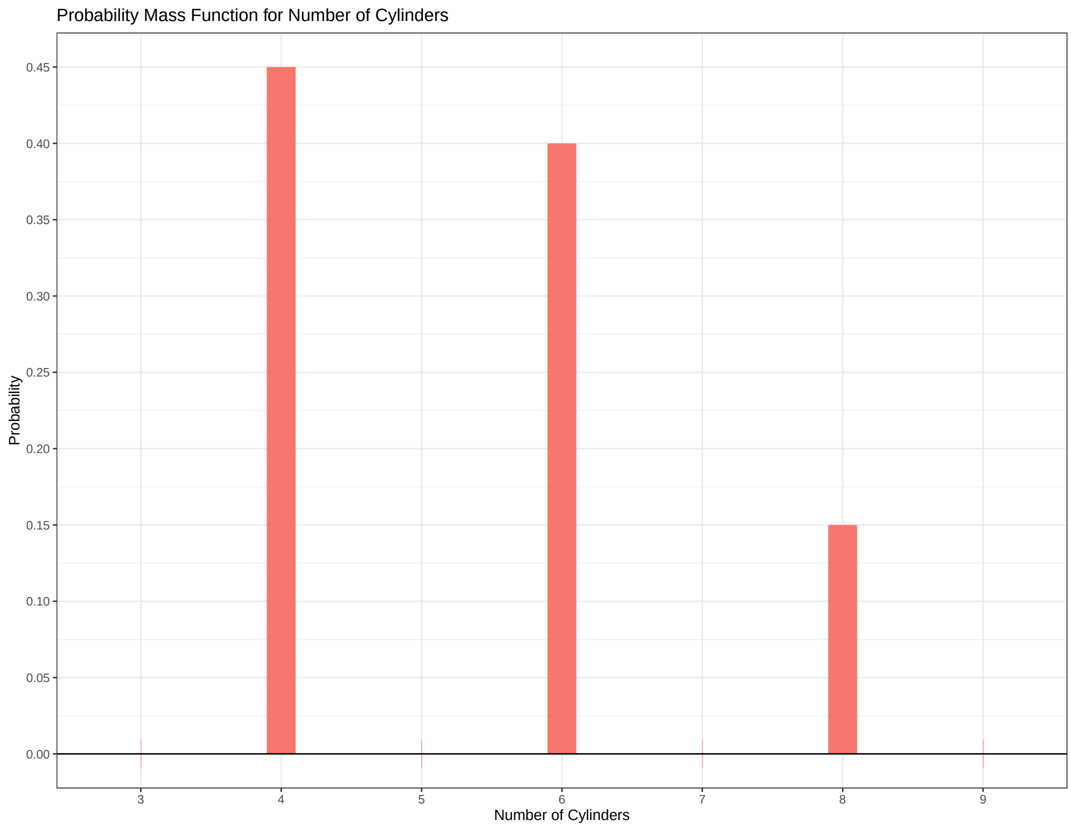
(ii) more than 4 cylinders?

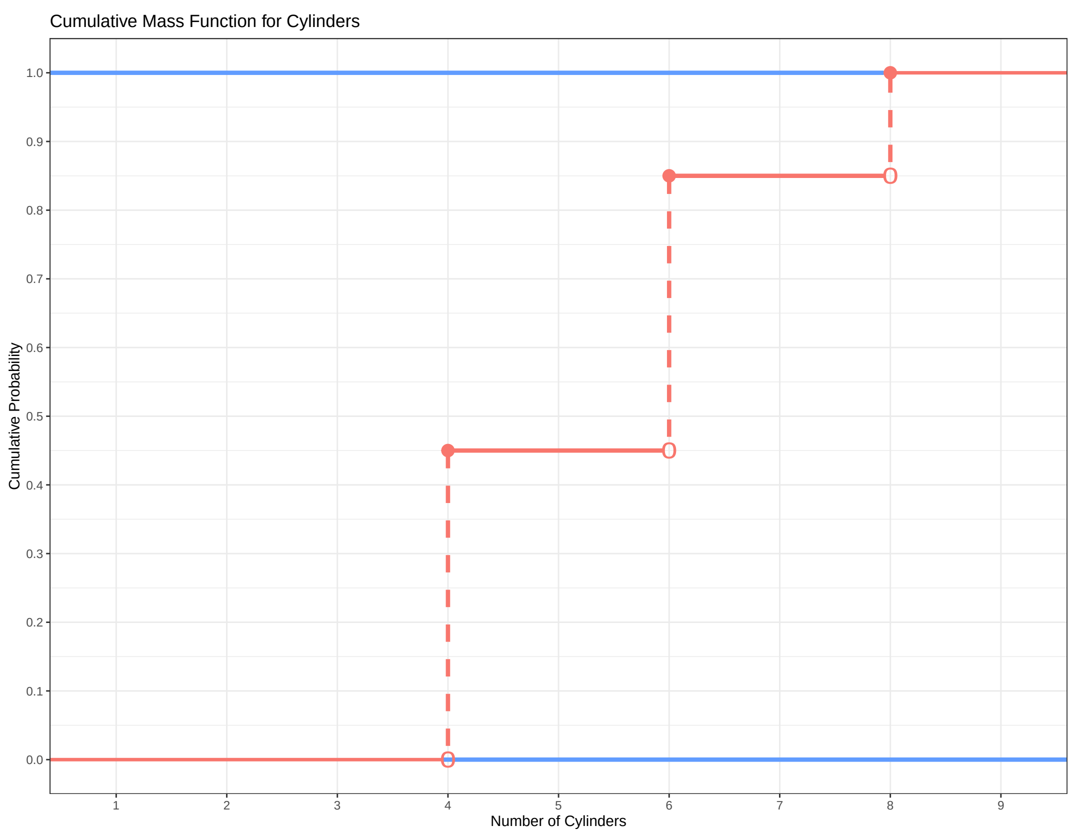
Method 1: Using pmf

$$\begin{aligned} \mathbf{P}(X > 4) &= \mathbf{P}(X = 5) + \mathbf{P}(X = 6) + \mathbf{P}(X = 7) + \mathbf{P}(X = 8) \\ &= 0 + 0.40 + 0 + 0.15 \\ &= 0.55 \end{aligned}$$

Method 2: Using cmf

$$\begin{aligned} \mathbf{P}(X > 4) &= 1 - \mathbf{P}(X \leq 4) \\ &= 1 - 0.45 \\ &= 0.55 \end{aligned}$$


Probability Mass Function for Number of Cylinders
Probability
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
3
4
5
6
7
8
9
Number of Cylinders


Cumulative Mass Function for Cylinders
Cumulative Probability
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
1
2
3
4
5
6
7
8
9
Number of Cylinders