Tutorial 2

January 30, 2020

Question 1

Let A, B be two events. We assume $\mathbf{P}(B) > 0$. Show that $\mathbf{P}(A \mid B) + \mathbf{P}(A^c \mid B) = 1$.

We first note that $A \cap B$ and $A^c \cap B$ are disjoint events and $(A \cap B) \cup (A^c \cap B) = B$.

$$\begin{aligned} \mathbf{P}(A \mid B) + \mathbf{P}(A^c \mid B) &= \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} + \frac{\mathbf{P}(A^c \cap B)}{\mathbf{P}(B)} \\ &= \frac{\mathbf{P}((A \cap B) \cup (A^c \cap B))}{\mathbf{P}(B)} \\ &= \frac{\mathbf{P}(B)}{\mathbf{P}(B)} \\ &= 1, \quad \text{provided that } \mathbf{P}(B) > 0 \end{aligned}$$

Question 2

70% of the light aircraft that disappear while in flight in a certain country are subsequently discovered. Of the aircraft that are discovered, 60% have an emergency locator, whereas 90% of the aircraft not discovered do not have such a locator. Suppose a light aircraft has disappeared.

Let D be the event an aircraft is discovered, and D^c be the event an aircraft is not discovered.

Let L be the event an aircraft has a locator, and L^c be the event an aircraft does not have a locator.

- *"70% of aircraft that disappear are discovered"* tells us that $\mathbf{P}(D) = 0.7$.
  This also tells us that $\mathbf{P}(D^c) = 1 - 0.7 = 0.3$.
- *"Of the aircraft discovered, 60% have an emergency locator"* tells us that $\mathbf{P}(L \mid D) = 0.6$.
  From the results of **Question 1**, this also tells us that $\mathbf{P}(L^c \mid D) = 0.4$.
- *"90% of aircraft not discovered do not have such a locator"* tells us that $\mathbf{P}(L^c \mid D^c) = 0.9$.
  From the results of **Question 1**, this also tells us that $\mathbf{P}(L \mid D^c) = 0.1$.

(a) If it has an emergency locator, what is the probability that it will not be discovered?

The probability that we seek is:

$$\mathbf{P}(D^c \mid L) = \frac{\mathbf{P}(D^c \cap L)}{\mathbf{P}(L)} \quad \begin{matrix} \leftarrow \text{unknown} \\ \leftarrow \text{unknown} \end{matrix}$$

By the law of total probability:

$$\begin{aligned}\mathbf{P}(L) &= \mathbf{P}(L \cap D) + \mathbf{P}(L \cap D^c) \\ &= \mathbf{P}(L \mid D)\,\mathbf{P}(D) + \mathbf{P}(L \mid D^c)\,\mathbf{P}(D^c) \\ &= (0.6)(0.7) + (0.1)(0.3) \\ &= 0.45\end{aligned}$$

We note that $\mathbf{P}(D^c \cap L)$ is the same as $\mathbf{P}(L \cap D^c)$, which we found when computing $\mathbf{P}(L)$ in the previous step. Therefore:

$$\mathbf{P}(D^c \mid L) = \frac{\mathbf{P}(D^c \cap L)}{\mathbf{P}(L)} = \frac{(0.1)(0.3)}{0.45} = \frac{1}{15} \approx 0.0\overline{66}$$

(b) If it does not have a locator, what is the probability it will be discovered?

The probability that we seek is:

$$\mathbf{P}(D \mid L^c) = \frac{\mathbf{P}(D \cap L^c)}{\mathbf{P}(L^c)} \quad \begin{array}{l}\leftarrow \text{unknown} \\ \leftarrow \text{unknown}\end{array}$$

Since we calculated $\mathbf{P}(L)$ in (a), its complement can be found as:

$$\mathbf{P}(L^c) = 1 - \mathbf{P}(L) = 1 - 0.45 = 0.55$$

From the given information that $\mathbf{P}(L^c \mid D) = 0.4$, we can solve for $\mathbf{P}(D \cap L^c)$ by recalling that:

$$\begin{aligned}\mathbf{P}(L^c \mid D) &= \frac{\mathbf{P}(L^c \cap D)}{\mathbf{P}(D)} \\ &= \frac{\mathbf{P}(D \cap L^c)}{\mathbf{P}(D)}\end{aligned}$$

Rearranging, we obtain:

$$\mathbf{P}(D \cap L^c) = \mathbf{P}(L^c \mid D)\,\mathbf{P}(D) = (0.4)(0.7)$$

Then:

$$\mathbf{P}(D \mid L^c) = \frac{\mathbf{P}(D \cap L^c)}{\mathbf{P}(L^c)} = \frac{(0.4)(0.7)}{0.55} = \frac{28}{55} \approx 0.509$$

Question 3

An aircraft seam requires 25 rivets. The seam will have to be reworked if any of these rivets are defective. Suppose rivets are defective independently of one another, each with the same probability.

Let X be the number of defective rivets on a seam.
Let p be the probability a rivet is defective.
Since a seam requires 25 rivets, $X \sim \text{Binomial}(n = 25,\, p)$.

(a) If 20% of all seams need reworking, what is the probability that a rivet is defective?

If 20% of seams need reworking, and reworking occurs when even one rivet is defective, this tells us that:

$$
\begin{aligned}
0.2 &= \mathbf{P}\left(X = 1\right) + \mathbf{P}\left(X = 2\right) + \ldots + \mathbf{P}\left(X = 25\right) \\
&= 1 - \mathbf{P}\left(X = 0\right) \\
&= 1 - \underbrace{\binom{25}{0}}_{=1} \underbrace{p^0}_{=1} (1-p)^{25}
\end{aligned}
$$

Rearranging to solve for p:

$$
\begin{aligned}
(1-p)^{25} &= 1 - 0.2 \\
(1-p)^{25} &= 0.8 \\
1 - p &= \sqrt[25]{0.8} \\
p &= 1 - \sqrt[25]{0.8} \\
p &\approx 0.00889
\end{aligned}
$$

(b) How small should the probability of a defective rivet be to ensure that only 10% of all seams need reworking?

By following the exact same reasoning and steps as (a), we obtain:

$$
p = 1 - \sqrt[25]{0.9} \approx 0.00421
$$

Question 4

A chemical supply company currently has in stock 100 lb of a certain chemical, which it sells to customers in 5 lb batches. Let X be the number of batches ordered by a randomly selected customer, and suppose that X has the following pmf:

$$p(x) = \mathbf{P}(X = x) = \begin{cases} 0.2 & x = 1 \\ 0.4 & x = 2 \\ 0.3 & x = 3 \\ 0.1 & x = 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute $\mathbf{E}(X)$ and $\mathbf{Var}(X)$.

$$\begin{aligned}
\mathbf{E}(X) &= \sum_{x \in S_X} x \cdot p(x) \\
&= \sum_{i=1}^{4} x_i \cdot p(x_i) \\
&= (1)(0.2) + (2)(0.4) + (3)(0.3) + (4)(0.1) \\
&= 2.3 \\
\mathbf{E}\left(X^2\right) &= \sum_{x \in S_X} x^2 \cdot p(x) \\
&= \sum_{i=1}^{4} x_i^2 \cdot p(x_i) \\
&= (1^2)(0.2) + (2^2)(0.4) + (3^2)(0.3) + (4^2)(0.1) \\
&= 6.1 \\
\mathbf{Var}(X) &= \mathbf{E}\left(X^2\right) - (\mathbf{E}(X))^2 \\
&= 6.1 - (2.3)^2 \\
&= 0.81
\end{aligned}$$

(b) Compute the expected number of batches left after the next customer's order is shipped. [Hint: The number of pounds left is a linear function of X].

Since we start with 100 lb and each batch is 5 lb, we can model the number of pounds left after the next customer's order as:

$$Y = 100 - 5X$$

Applying the properties of the expected value operator, the expected number of pounds left is:

$$\mathbf{E}(Y) = \mathbf{E}(100 - 5X) = 100 - 5\mathbf{E}(X) = 100 - 5(2.3) = 88.5$$