Tutorial 7

March 12, 2020

Question 1

Each front tire on a particular type of vehicle is supposed to be filled to a pressure of 26 psi. Suppose the actual air pressure in each tire is a random variable X for right tire and Y for left one, with joint pdf:

$$f(x, y) = \begin{cases} K(x^2 + y^2) & 20 \leq x,\ y \leq 30 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find K.

If we integrate the joint pdf over $\mathbb{R}^2$ we should get 1. This reduces to integrating over $20 \leq x \leq 30$ and $20 \leq y \leq 30$.

$$K \int_{20}^{30} \int_{20}^{30} x^2 + y^2 \, dx \, dy = K \int_{20}^{30} \frac{1}{3}x^3 + xy^2 \Big|_{x=20}^{x=30} dy$$

$$= K \int_{20}^{30} \frac{19000}{3} + 10y^2 \, dy$$

$$= K \left(\frac{19000}{3} y + \frac{10}{3} y^3 \right) \Big|_{y=20}^{y=30}$$

$$= K \left(\frac{190000}{3} + \frac{190000}{3} \right)$$

$$= K \cdot \frac{380000}{3}$$

$$K \cdot \frac{380000}{3} = 1 \quad \Longrightarrow \quad K = \frac{3}{380000} \approx 7.89 \times 10^{-6}$$

Moving forward, we will just leave K as is since its numerical form is too long to keep rewriting.

(b) Determine the marginal distribution of air pressure in the right tire and in the left tire.

To find the marginal distribution of X we integrate over Y. Similarly, to find the marginal distribution for Y, we integrate over X.

$$f_X(x) = K \int_{20}^{30} x^2 + y^2 \, dy$$

$$= K \left(x^2 y + \frac{1}{3} y^3 \right) \Big|_{y=20}^{y=30}$$

$$= K\left(10x^2 + \frac{19000}{3}\right)$$

The marginal distribution of X is:

$$f_X(x) = \begin{cases} K\left(10x^2 + \frac{19000}{3}\right) & 20 \le x \le 30 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, the marginal distribution of Y is:

$$f_Y(y) = \begin{cases} K\left(10y^2 + \frac{19000}{3}\right) & 20 \le y \le 30 \\ 0 & \text{otherwise} \end{cases}$$

(c) Compute the correlation coefficient between X and Y.

It should be noted that for $g(X)$, a function solely of X:

$$\mathbf{E}\left(g(X)\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\, f(x,y)\, dy\, dx = \int_{-\infty}^{\infty} g(x)\left(\int_{-\infty}^{\infty} f(x,y)\, dy\right) dx = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

A similar result applies for $h(Y)$, a function solely of Y.

$$\mathbf{E}\left(X\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= K\int_{20}^{30} 10x^3 + \frac{19000}{3}x\, dx$$

$$= K\left(\frac{10}{4}x^4 + \frac{19000}{6}x^2\right)\Bigg|_{x=20}^{x=30}$$

$$= K \cdot \frac{9625000}{3}$$

$$= \frac{1925}{76}$$

Similarly, $\mathbf{E}\left(Y\right) = \frac{1925}{76}$.

$$
\begin{aligned}
\mathbf{E}(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dy\, dx \\
&= K\int_{20}^{30}\int_{20}^{30} xy\left(x^2+y^2\right) dy\, dx \\
&= K\int_{20}^{30}\int_{20}^{30} x^3y + xy^3\, dy\, dx \\
&= K\int_{20}^{30} \frac{1}{2}x^3y^2 + \frac{1}{4}xy^4\Big|_{y=20}^{y=30} dx \\
&= K\int_{20}^{30} \frac{500}{2}x^3 + \frac{650000}{4}x\, dx \\
&= K\left(\frac{500}{8}x^4 + \frac{650000}{8}x^2\right)\Big|_{x=20}^{x=30} \\
&= K\cdot 81250000 \\
&= \frac{24375}{38}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \\
&= K\int_{20}^{30} x^2\left(10x^2 + \frac{19000}{3}\right) dx \\
&= K\int_{20}^{30} 10x^4 + \frac{19000}{3}x^2\, dx \\
&= K\left(\frac{10}{5}x^5 + \frac{19000}{9}x^3\right)\Big|_{x=20}^{x=30} \\
&= K\cdot\frac{740800000}{9} \\
&= \frac{37040}{57}
\end{aligned}
$$

Similarly, $\mathbf{E}\left(Y^2\right) = \dfrac{37040}{57}$.

$$\begin{aligned}\mathbf{Var}\left(X\right) &= \mathbf{E}\left(X^2\right) - \left(\mathbf{E}\left(X\right)\right)^2 \\ &= \frac{37040}{57} - \left(\frac{1925}{76}\right)^2\end{aligned}$$

Similarly, $\mathbf{Var}\left(Y\right) = \dfrac{37040}{57} - \left(\dfrac{1925}{76}\right)^2$.

$$\begin{aligned}\mathbf{Corr}\left(X,Y\right) &= \frac{\mathbf{Cov}\left(X,Y\right)}{\sqrt{\mathbf{Var}\left(X\right)\mathbf{Var}\left(Y\right)}} \\ &= \frac{\mathbf{E}\left(XY\right) - \mathbf{E}\left(X\right)\mathbf{E}\left(Y\right)}{\mathbf{Var}\left(X\right)} && \text{[Since } \mathbf{Var}\left(X\right) = \mathbf{Var}\left(Y\right)\text{]} \\ &= \frac{\dfrac{24375}{38} - \left(\dfrac{1925}{76}\right)\left(\dfrac{1925}{76}\right)}{\dfrac{34070}{57} - \left(\dfrac{1925}{76}\right)^2} \\ &= -0.013085\end{aligned}$$

(d) Are X and Y independent?

By inspection, $f(x,y) \neq f_X(x) \cdot f_Y(y) \quad \forall \quad (x,y) \in \mathbb{R}^2$. X and Y are not independent.

Question 2

Annie and Alvin have agreed to meet for lunch between 12:00 pm and 1:00 pm. Denote Annie's arrival time by X and Alvin's arrival time by Y. Suppose X and Y are independent with pdfs:

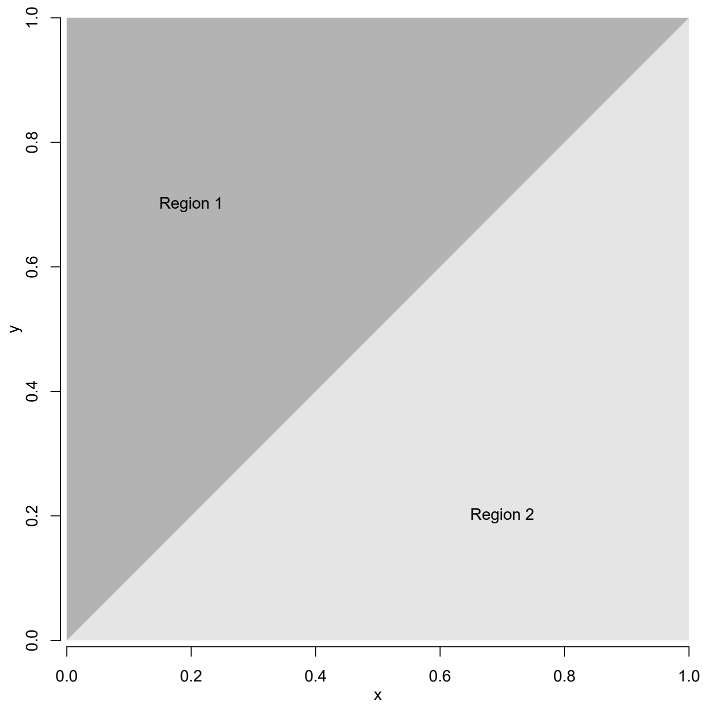
$$f_X(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 2y & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the average waiting time for the person who arrives first? [Hint: Consider $h(x,y) = |x-y|$.]

Recall that the absolute value function is defined as:

$$|x-y| \quad = \quad \begin{cases} x-y & x-y \geq 0 \\ -(x-y) & x-y < 0 \end{cases} \quad = \quad \begin{cases} x-y & x \geq y \\ -(x-y) & x < y \end{cases}$$

Before beginning any integration we need to graph $|x-y|$ and identify the regions where the function changes sign.



- Since $0 \le x \le 1$ and $0 \le y \le 1$, this means that we are contained in a square in the first quadrant.
- In Region 1 (R_1), $x < y$ and $|x - y| = -(x - y)$.
- In Region 2 (R_2), $x > y$ and $|x - y| = x - y$.

Therefore:

$$\mathbf{E}\left(|X - Y|\right) = \iint\limits_{R_1 \cup R_2} |x - y|\, 6x^2 y \, dy \, dx = \iint\limits_{R_1} |x - y|\, 6x^2 y \, dy \, dx + \iint\limits_{R_2} |x - y|\, 6x^2 y \, dy \, dx$$

$$\begin{aligned}
\iint\limits_{R_1} |x - y|\, 6x^2 y \, dy \, dx &= -6 \int_0^1 \int_x^1 (x - y) x^2 y \, dy \, dx \\
&= -6 \int_0^1 \int_x^1 x^3 y - x^2 y^2 \, dy \, dx \\
&= -6 \int_0^1 \frac{1}{2} x^3 y^2 - \frac{1}{3} x^2 y^3 \Big|_{y=x}^{y=1} dx \\
&= -\frac{6}{6} \int_0^1 -x^5 + 3x^3 - 2x^2 \, dx \\
&= -\left(-\frac{1}{6} x^6 + \frac{3}{4} x^4 - \frac{2}{3} x^3 \right) \Big|_{x=0}^{x=1}
\end{aligned}$$

$$= -\left(-\frac{1}{6} + \frac{3}{4} - \frac{2}{3}\right)$$

$$= \frac{1}{12}$$

$$\iint_{R_2} |x-y|\, 6x^2y\, dy\, dx = 6\int_0^1 \int_0^x (x-y)x^2y\, dy\, dx$$

$$= 6\int_0^1 \int_0^x x^3y - x^2y^2\, dy\, dx$$

$$= 6\int_0^1 \frac{1}{2}x^3y^2 - \frac{1}{3}x^2y^3\Big|_{y=0}^{y=x} dx$$

$$= \frac{6}{6}\int_0^1 x^5\, dx$$

$$= \frac{1}{6}x^6\Big|_{x=0}^{x=1}$$

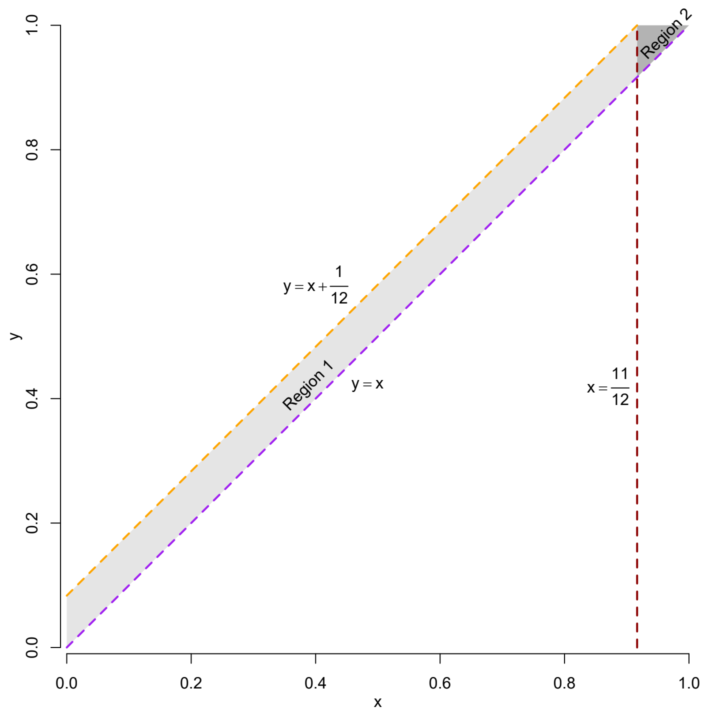
$$= \frac{1}{6}$$

$$\mathbf{E}(|X-Y|) = \iint_{R_1 \cup R_2} |x-y|\, 6x^2y\, dy\, dx = \frac{1}{12} + \frac{1}{6} = \frac{3}{12} = \frac{1}{4}$$

(b) What is the probability that Annie will have to wait up to 5 minutes for Alvin?

Since the pdfs above were given in hours, 5 minutes is 1/12 hours. In order to compute this integral we will need to to first graph the region of interest. From the question, we have that Annie arrives first and that Alvin will arrive up to 5 minutes later. The region of interest would then be the region between the lines $y = x$ and $y = x + 1/12$, while also keeping in mind that $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

From the graph below, we can see that we will need to break this region up into two parts.



Let E represent the event that Annie waits up to 5 minutes for Alvin. Denoting Region 1 as R_1 and Region 2 as R_2,

$$P(E) = \iint_{R_1 \cup R_2} 6x^2 y \, dy \, dx = \iint_{R_1} 6x^2 y \, dy \, dx + \iint_{R_2} 6x^2 y \, dy \, dx$$

$$\iint_{R_1} 6x^2 y \, dy \, dx = \int_0^{\frac{11}{12}} \int_x^{x+\frac{1}{12}} 6x^2 y \, dy \, dx$$

$$= \int_0^{\frac{11}{12}} 3x^2 y^2 \Big|_{y=x}^{y=x+\frac{1}{12}} dx$$

$$= \int_0^{\frac{11}{12}} \frac{3}{6} x^3 + \frac{3}{144} x^2 \, dx$$

$$= \left(\frac{3}{24} x^4 + \frac{1}{144} x^3 \right) \Big|_{x=0}^{x=\frac{11}{12}}$$

$$= \left(\frac{3}{24} \right) \left(\frac{11}{12} \right)^4 + \left(\frac{1}{144} \right) \left(\frac{11}{12} \right)^3$$

$$\approx 0.093607$$

$$\iint_{R_2} 6x^2 y \, dy \, dx = \int_{\frac{11}{12}}^1 \int_x^1 6x^2 y \, dy \, dx$$

$$= \int_{\frac{11}{12}}^{1} 3x^2y^2 \Big|_{y=x}^{y=1} dx$$

$$= \int_{\frac{11}{12}}^{1} -3x^4 + 3x^2 \, dx$$

$$= -\frac{3}{5}x^5 + x^3 \Big|_{x=\frac{11}{12}}^{x=1}$$

$$= -\frac{3}{5}\left(1 - \left(\frac{11}{12}\right)^5\right) + \left(1 - \left(\frac{11}{12}\right)^3\right)$$

$$\approx 0.018082$$

In conclusion,

$$P(E) = \left(\frac{3}{24}\right)\left(\frac{11}{12}\right)^4 + \left(\frac{1}{144}\right)\left(\frac{11}{12}\right)^3 - \frac{3}{5}\left(1 - \left(\frac{11}{12}\right)^5\right) + \left(1 - \left(\frac{11}{12}\right)^3\right) = 0.111689$$