Tutorial 8

March 19, 2020

Question 1

The inside diameter of a randomly selected piston ring is a random variable with mean 12cm and standard deviation 0.4cm. If a random sample of 32 rings are selected, what is the probability that their average diameter will be between 11.9 and 12.1?

Let X represent the diameter of a piston ring. From the question, it given that:

$$\mathbf{E}(X) = 12, \quad \mathbf{Var}(X) = 0.4^2, \quad n = 32$$

Since $n > 30$, by Central Limit Theorem,

$$\frac{\overline{X} - 12}{0.4/\sqrt{32}} \stackrel{\cdot}{\sim} N(0, 1)$$

$$\begin{aligned}
\mathbf{P}\left(11.9 \leq \overline{X} \leq 12.1\right) &= \mathbf{P}\left(\overline{X} \leq 12.1\right) - \mathbf{P}\left(\overline{X} \leq 11.9\right) \\
&\approx \mathbf{P}\left(\frac{\overline{X} - 12}{0.4/\sqrt{32}} \leq \frac{12.1 - 12}{0.4/\sqrt{32}}\right) - \mathbf{P}\left(\frac{\overline{X} - 12}{0.4/\sqrt{32}} \leq \frac{11.9 - 12}{0.4/\sqrt{32}}\right) \\
&= \mathbf{P}(Z \leq 1.4142) - \mathbf{P}(Z \leq -1.4142) \\
&= 0.8427
\end{aligned}$$

Question 2

Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution over $[0, \theta]$. Let

$$\widehat{\theta}_1 = Y = \max(X_1, \ldots, X_n).$$

(a) Compute the cdf of Y and conclude that the pdf of Y is

$$f(y) = \begin{cases} \frac{ny^{n-1}}{\theta^n} & 0 \leq y \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

First note that $X_i \sim \text{Uniform}(0, \theta)$, each with cdf given by:

$$F_i(x) = \mathbf{P}(X_i \le x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{\theta} & 0 \le x \le \theta \\ 1 & x > \theta \end{cases}$$

for $i = 1, \ldots, n$. Then the cdf of Y is:

$$\begin{aligned} F(y) &= \mathbf{P}(Y \le y) && \text{(Definition of cdf)} \\ &= \mathbf{P}(\max(X_1, \ldots, X_n) \le y) \end{aligned}$$

If the maximum of a sample is less than or equal to some value y, then *all* elements of the sample are less than or equal to the value y.

$$\begin{aligned} &= \mathbf{P}((X_1 \le y) \cap \ldots \cap (X_n \le y)) \\ &= \mathbf{P}(X_1 \le y) \cdot \ldots \cdot \mathbf{P}(X_n \le y) && \text{(Independence)} \\ &= F_1(y) \cdot \ldots \cdot F_n(y) \\ &= (F_i(y))^n && \text{(Identically distributed)} \\ &= \begin{cases} 0 & y < 0 \\ \left(\dfrac{y}{\theta}\right)^n & 0 \le y \le \theta \\ 1 & y > \theta \end{cases} \end{aligned}$$

The pdf of Y is obtained as usual, by differentiating the cdf (with respect to y).

$$\begin{aligned} f(y) &= F'(y) \\ &= n\left(\frac{y}{\theta}\right)^{n-1} \cdot \left(\frac{1}{\theta}\right) \\ &= \begin{cases} \dfrac{ny^{n-1}}{\theta^n} & 0 \le y \le \theta \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(b) What is the pdf of $Y = \max(X_1, X_2, X_3, X_4)$?

With $n = 4$,

$$f(y) = \begin{cases} \dfrac{4y^3}{\theta^4} & 0 \le y \le \theta \\ 0 & \text{otherwise} \end{cases}$$

(c) Show that $\widehat{\theta}_1$ is biased, but that $\widehat{\theta}_2 = \frac{n+1}{n} Y$ is unbiased.

We want to show that $\mathbf{E}\,(\widehat{\theta}_1)$ is biased, i.e. $\mathbf{E}\,(\widehat{\theta}_1) \neq \theta$. Note that the distribution of $\widehat{\theta}_1$ is the distribution of Y, found in (a).

$$\begin{aligned}
\mathbf{E}\,(\widehat{\theta}_1) &= \mathbf{E}\,(Y) \\
&= \int_{-\infty}^{\infty} y\, f(y)\, dy \\
&= \int_{0}^{\theta} y\, \frac{n y^{n-1}}{\theta^n}\, dy \\
&= \int_{0}^{\theta} \frac{n}{\theta^n}\, y^n\, dy \\
&= \frac{n}{n+1} \cdot \frac{1}{\theta^n} \cdot y^{n+1} \Big|_{y=0}^{y=\theta} \\
&= \frac{n}{n+1} \cdot \frac{\theta^{n+1}}{\theta^n} \\
&= \frac{n}{n+1} \cdot \theta
\end{aligned}$$

As $\mathbf{E}\,(\widehat{\theta}_1) \neq \theta$, $\widehat{\theta}_1$ is a biased estimator of θ.

To show that $\widehat{\theta}_2 = \frac{n+1}{n} Y$ is an unbiased estimator of θ, we want to show that $\mathbf{E}\,(\widehat{\theta}_2) = \theta$. As we have already calculated $\mathbf{E}\,(Y)$ (above), we do not need to calculate it again and can simply use the properties of the expectation operator.

$$\mathbf{E}(\widehat{\theta}_2) = \mathbf{E}\left(\frac{n+1}{n} Y\right) = \frac{n+1}{n} \mathbf{E}\,(Y) = \frac{n+1}{n} \cdot \frac{n}{n+1} \cdot \theta = \theta$$

Since $\mathbf{E}\,(\widehat{\theta}_2) = \theta$, $\widehat{\theta}_2$ is an unbiased estimator of θ.

Question 3

Let X be the proportion of allotted time a randomly selected student spends working on a certain aptitude test. Suppose X has pdf:

$$f(x) = \begin{cases} (\theta + 1)\, x^{\theta} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > -1$ is an unknown parameter. Suppose a random sample of ten students yielded the following data:

0.92 0.79 0.90 0.65 0.86 0.47 0.73 0.97 0.94 0.77

(a) Find the method of moments estimator for θ and compute the corresponding estimate for the given data.

To obtain the method of moments estimator, we need to set the first population moment equal to the first sample moment. Therefore, we should start by finding the first population moment, $\mathbf{E}(X)$.

$$\begin{aligned}
\mathbf{E}(X) &= \int_{-\infty}^{\infty} x\, f(x)\, dx \\
&= \int_{0}^{1} x\,(\theta+1)\, x^{\theta}\, dx \\
&= \int_{0}^{1} (\theta+1)\, x^{\theta+1}\, dx \\
&= \frac{\theta+1}{\theta+2} \cdot x^{\theta+2}\Big|_{x=0}^{x=1} \\
&= \frac{\theta+1}{\theta+2}
\end{aligned}$$

Now, we can equate the first population moment, $\mathbf{E}(X)$, with the first sample moment, $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and solve for our method of moments estimator, $\widehat{\theta}$.

$$\mathbf{E}(X) = \overline{X} \quad \Longleftrightarrow \quad \frac{\widehat{\theta}+1}{\widehat{\theta}+2} = \overline{X} \quad \Longleftrightarrow \quad \widehat{\theta} = \frac{2\overline{X}-1}{1-\overline{X}} = \frac{\overline{X}}{1-\overline{X}} - 1$$

Using the given data, $\overline{x} = 0.8$. The estimate for θ is:

$$\widehat{\theta} = \frac{\overline{x}}{1-\overline{x}} - 1 = \frac{0.8}{1-0.8} - 1 = 3$$

(b) Find the maximum likelihood estimator for θ and compute the corresponding estimate for the given data.

We start by obtaining the likelihood function. The likelihood function is the same as the joint density, but is a function of θ rather the x_is.

$$\begin{aligned}
\mathcal{L}(\theta;\, x_1,\, \ldots,\, x_n) &= f(x_1,\, \ldots,\, x_n;\, \theta) \\
&= \prod_{i=1}^{n} f(x_i;\, \theta) && \text{(Due to independence)} \\
&= \prod_{i=1}^{n} (\theta+1)\, x_i^{\theta} \\
&= (\theta+1)^n \prod_{i=1}^{n} x_i^{\theta}
\end{aligned}$$

$$= (\theta + 1)^n \left(\prod_{i=1}^{n} x_i \right)^{\theta}$$

It is often easier to work with the log-likelihood, which is obtained by taking the (natural) logarithm of the likelihood function.

$$\begin{aligned} \ell(\theta;\, x_1,\, \ldots,\, x_n) &= \ln\left(\mathcal{L}(\theta;\, x_1,\, \ldots,\, x_n)\right) \\ &= n \ln(\theta + 1) \,+\, \theta \ln\left(\prod_{i=1}^{n} x_i \right) \end{aligned}$$

I will not apply logarithm properties to rewrite $\ln\left(\prod_{i=1}^{n} x_i\right)$ as $\sum_{i=1}^{n} \ln(x_i)$ since the former is easier to punch into a calculator than the latter.

From elementary calculus, the extrema occur where the first derivative is equal to zero.

$$\frac{d\ell}{d\theta} = \frac{n}{\theta + 1} \,+\, \ln\left(\prod_{i=1}^{n} x_i \right)$$

$$\begin{aligned} & \left.\frac{d\ell}{d\theta}\right|_{\theta = \widehat{\theta}} &= \quad & 0 \\ \iff \quad & \frac{n}{\widehat{\theta} + 1} + \ln\left(\prod_{i=1}^{n} x_i \right) &= \quad & 0 \\ \iff \quad & \widehat{\theta} + 1 &= \quad & \frac{-n}{\ln\left(\prod_{i=1}^{n} x_i \right)} \\ \iff \quad & \widehat{\theta} &= \quad & \frac{-n}{\ln\left(\prod_{i=1}^{n} x_i \right)} \,-\, 1 \end{aligned}$$

To verify that $\ell(\widehat{\theta})$ is indeed a maximum, we need to ensure that the second derivative is less than zero (concave-down) at $\widehat{\theta}$.

$$\frac{d^2\ell}{d\theta^2} = -\frac{n}{(\theta + 1)^2}$$

In this case, we don't actually need to evaluate the second derivative at the point $\widehat{\theta}$ since it can be seen that the second derivative is less than zero for all values of $\theta \in \Theta = \{\theta : \theta > -1\}$. Therefore,

$$\widehat{\theta} = \frac{-n}{\ln\left(\prod_{i=1}^{n} X_i \right)} \,-\, 1$$

is the MLE for θ. Using the given data, $\ln\left(\prod_{i=1}^{n} x_i\right) \approx -2.4295$. The estimate for θ is:

$$\widehat{\theta} \,=\, \frac{-10}{-2.4295} \,-\, 1 \,=\, 3.116$$