Tutorial 9 Part (c)

Recall that if

$$X \sim \text{Gamma}(2,\, \theta), \quad \text{then} \quad Y := \frac{X}{\theta} \sim \text{Gamma}(2,\, 1)$$

If we solve for the value $\tilde{\mu}$ such that $F_Y(\tilde{\mu}) = 1/2$, then

$$F_Y(\tilde{\mu}) = \mathbf{P}\left(Y \leq \tilde{\mu}\right) = \mathbf{P}\left(\frac{X}{\theta} \leq \tilde{\mu}\right) = \mathbf{P}\left(X \leq \theta\tilde{\mu}\right) = F_X(\theta\tilde{\mu}) = 1/2 \tag{1}$$

In other words, if $X \sim \text{Gamma}(2,\, \theta)$, given $\tilde{\mu}$, the median of F_X is $\theta\tilde{\mu}$.

We first define the function `GY` to represent

$$G(y) = F_Y(y) - 1/2 = 1 - (1 + y)\, e^{-y} - 1/2 = 0$$

```r
GY <- function(y) {
  1 - (1 + y) * exp(-y) - 1/2
}
```

We will search in the interval $(0, 5)$ since the original support was $x > 0$, and as a result, $y > 0$.

```r
uniroot(GY, interval=c(0, 5))
```

```
## $root
## [1] 1.678329
##
## $f.root
## [1] -5.562972e-06
##
## $iter
## [1] 6
##
## $init.it
## [1] NA
##
## $estim.prec
## [1] 6.103516e-05
```

The median of $F_Y(y)$ is 1.678. Then the median of $F_X(x)$ is $\theta * 1.678$. We can extract the value of the root and store it in the variable `mu_tilde`.

```r
mu_tilde <- uniroot(GY, interval=c(0, 5))$root
```

To check the claim in (1), let's try the values $\theta = 0.25$, $\theta = 1$, and $\theta = 3$ (the `pgamma` function returns $\mathbf{P}\left(X \leq x\right)$).

```r
pgamma(0.25 * mu_tilde, shape=2, scale=0.25)
```

```
## [1] 0.4999944
```

```r
pgamma(mu_tilde, shape=2, scale=1)
```

```
## [1] 0.4999944
```

```r
pgamma(3 * mu_tilde, shape=2, scale=3)
```

```
## [1] 0.4999944
```