Tutorial 9

March 26, 2020

Question 1

Consider a random sample from a Gamma$(2, \theta)$ distribution:

$$f(x;\,\theta) = \begin{cases} \theta^{-2} x e^{-x/\theta} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where θ is unknown.

(a) Find the MLE of θ. Check if it is unbiased.

We start by obtaining the likelihood function. The likelihood function is the same as the joint density, but is a function of θ rather than the x_is.

$$\begin{aligned}
\mathcal{L}(\theta;\, x_1,\, \ldots,\, x_n) &= f(x_1,\, \ldots,\, x_n;\, \theta) \\
&= \prod_{i=1}^{n} f(x_i;\, \theta) \\
&= \prod_{i=1}^{n} \theta^{-2}\, x_i\, \exp\{-x_i / \theta\} \\
&= \theta^{-2n} \left(\prod_{i=1}^{n} x_i \right) \exp\left\{ -\frac{\sum_{i=1}^{n} x_i}{\theta} \right\}
\end{aligned}$$

It is often easier to work with the log-likelihood, which is obtained by taking the (natural) logarithm of the likelihood function.

$$\begin{aligned}
\ell(\theta;\, x_1,\, \ldots,\, x_n) &= \ln\left(\mathcal{L}(\theta;\, x_1,\, \ldots,\, x_n)\right) \\
&= -2n \ln(\theta) + \ln\left(\prod_{i=1}^{n} x_i \right) - \frac{\sum_{i=1}^{n} x_i}{\theta}
\end{aligned}$$

From elementary calculus, the extrema occur where the first derivative is equal to zero.

$$\frac{d\ell}{d\theta} = -\frac{2n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2}$$

$$\begin{aligned}
\left.\frac{d\ell}{d\theta}\right|_{\theta=\widehat{\theta}} &= 0 \\
\iff \qquad -\frac{2n}{\widehat{\theta}} + \frac{\sum_{i=1}^{n} x_i}{\widehat{\theta}^2} &= 0
\end{aligned}$$

$$\begin{aligned}
\iff \quad \frac{\sum_{i=1}^{n} x_i}{\widehat{\theta}^2} &= \frac{2n}{\widehat{\theta}} \\
\iff \quad \sum_{i=1}^{n} x_i &= 2n\widehat{\theta} \\
\iff \quad \widehat{\theta} &= \frac{\sum_{i=1}^{n} x_i}{2n}
\end{aligned}$$

To verify that $\ell(\widehat{\theta})$ is indeed a maximum, we need to check that the second derivative is less than zero (concave-down) at $\widehat{\theta}$.

$$\frac{d^2\ell}{d\theta^2} = \frac{2n}{\theta^2} - \frac{2\sum_{i=1}^{n} x_i}{\theta^3}$$

$$\begin{aligned}
\left.\frac{d^2\ell}{d\theta^2}\right|_{\theta=\widehat{\theta}} &= \frac{2n}{\left(\frac{\sum_{i=1}^{n} x_i}{2n}\right)^2} - \frac{2\sum_{i=1}^{n} x_i}{\left(\frac{\sum_{i=1}^{n} x_i}{2n}\right)^3} \\
&= \frac{(2n)^3}{\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{2\,(2n)^3 \sum_{i=1}^{n} x_i}{\left(\sum_{i=1}^{n} x_i\right)^3} \\
&= \frac{(2n)^3}{\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{2\,(2n)^3}{\left(\sum_{i=1}^{n} x_i\right)^2} \\
&= -\frac{(2n)^3}{\left(\sum_{i=1}^{n} x_i\right)^2} < 0
\end{aligned}$$

Therefore,

$$\widehat{\theta} = \frac{\sum_{i=1}^{n} X_i}{2n} = \frac{\overline{X}}{2}$$

is the MLE for θ.

$\widehat{\theta}$ is an unbiased estimator of θ if $\mathbf{E}\,(\widehat{\theta}) = \theta$. Recall that the mean of a Gamma$(\alpha,\, \beta)$ distribution is $\alpha\beta$. Then for $X \sim$ Gamma$(2,\, \theta)$, $\mathbf{E}\,(X) = 2\theta$.

$$\mathbf{E}\,(\widehat{\theta}) = \mathbf{E}\left(\frac{\overline{X}}{2}\right) = \frac{1}{2}\mathbf{E}\,(\overline{X}) = \frac{1}{2}\,2\theta = \theta$$

As $\mathbf{E}\,(\widehat{\theta}) = \theta$, we conclude that $\widehat{\theta}$ is an unbiased estimator of θ.

(b) Find the cdf, F, of f.

$$\begin{aligned}
F(x;\, \theta) &= \int_{-\infty}^{x} f(t;\, \theta)\, dt \\
&= \int_{0}^{x} \theta^{-2}\, t\, e^{-t/\theta}\, dt
\end{aligned}$$

$$g(t) = t \qquad h(t) = -\theta\, e^{-t/\theta}$$

$$g'(t) = 1 \qquad h'(t) = e^{-t/\theta}$$

$$= \theta^{-2} \left[-\theta\, t\, e^{-t/\theta} \Big|_{t=0}^{t=x} + \theta \int_0^x e^{-t/\theta}\, dt \right]$$

$$= \theta^{-2} \left[-\theta\, t\, e^{-t/\theta} - \theta^2\, e^{-t/\theta} \right] \Big|_{t=0}^{t=x}$$

$$= \theta^{-2} \left[\left(-\theta\, x\, e^{-x/\theta} - \theta^2 e^{-x/\theta} \right) - \left(0 - \theta^2 \right) \right]$$

$$= \theta^{-2} \left[\left(-\theta x - \theta^2 \right) e^{-x/\theta} + \theta^2 \right]$$

$$= \left(-\frac{x}{\theta} - 1 \right) e^{-x/\theta} + 1$$

$$F(x;\, \theta) = \begin{cases} 0 & x \le 0 \\ 1 - \left(1 + \dfrac{x}{\theta} \right) e^{-x/\theta} & x > 0 \end{cases}$$

(c) Find the median of F. That is, the value $\widetilde{\mu}$ such that $F(\widetilde{\mu}) = 1/2$. **Hint:** since there is no closed form for the median, you will have to use numerical methods to invert F. I.e. Use software to solve $F(x) = 1/2$ or $F(x) - 1/2 = 0$.

Recall that if:

$$X \sim \text{Gamma}(2,\, \theta), \quad \text{then} \quad Y := \frac{X}{\theta} \sim \text{Gamma}(2,\, 1).$$

If we solve for the value $\widetilde{\mu}$ such that $F_Y(\widetilde{\mu}) = 1/2$, then

$$F_Y(\widetilde{\mu}) = \mathbf{P}\left(Y \le \widetilde{\mu}\right) = \mathbf{P}\left(\frac{X}{\theta} \le \widetilde{\mu}\right) = \mathbf{P}\left(X \le \theta\widetilde{\mu}\right) = F_X(\theta\widetilde{\mu}) = 1/2$$

In other words, if $X \sim \text{Gamma}(2,\, \theta)$, given $\widetilde{\mu}$, the median of F_X is $\theta\widetilde{\mu}$. Using **R**, it is found that $\widetilde{\mu} \approx 1.678$ (see supplementary file for code and output).

(d) Use the invariance principle for MLEs to compute the MLE of the median of F found in (c).

The invariance principle for MLEs states that if $\widehat{\theta}$ is a MLE for θ, then $g(\widehat{\theta})$ is a MLE for $g(\theta)$. As

$$\widehat{\theta} = \frac{\overline{X}}{2}$$

was a MLE for θ, then

$$g(\widehat{\theta}) = \widehat{\theta}\widetilde{\mu} = \frac{\overline{X}\widetilde{\mu}}{2}$$

is a MLE for $g(\theta) = \theta\widetilde{\mu}$.

(e) Suppose we observe the following sample: Compute the MLE of θ and the MLE of the median.

2.5 1.2 3.7 1.8 2.1 2.4

Using the given data, $\overline{x} = 2.283$.

An estimate for θ is:

$$\widehat{\theta} = \frac{\overline{x}}{2} = 2.283/2 = 1.1417$$

An estimate for $\theta\widetilde{\mu}$ is:

$$g(\widehat{\theta}) = \frac{\overline{x}\widetilde{\mu}}{2} = \frac{2.283 * 1.678}{2} = 1.9157$$