Tutorial 1

September 24, 2020

Question 1

Suppose we draw 2 cards (without replacement) from a deck of 52 cards.

Let $A =$ "the first card is an ace"
Let $B =$ "the second card is a spade"

Are A and B independent events?

A and B are independent events iff $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$. To verify this equality, we need to compute the individual probabilities.

$$\mathbf{P}(A) = 4/52 = 1/13$$

$$\begin{aligned}
\mathbf{P}(B) &= \mathbf{P}(\text{Spade} \rightarrow \text{Spade}) + \mathbf{P}(\text{Non-spade} \rightarrow \text{Spade}) \\
&= \frac{13 \cdot 12}{52 \cdot 51} + \frac{39 \cdot 13}{52 \cdot 51} \\
&= \frac{663}{2652} = \frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}(A \cap B) &= \mathbf{P}(\text{Ace of Non-spade} \rightarrow \text{Spade}) + \mathbf{P}(\text{Ace of Spades} \rightarrow \text{Spade}) \\
&= \frac{3 \cdot 13}{52 \cdot 51} + \frac{1 \cdot 12}{52 \cdot 51} \\
&= \frac{51}{52 \cdot 51} = \frac{1}{52}
\end{aligned}$$

$$\mathbf{P}(A) \cdot \mathbf{P}(B) = \frac{1}{13} \cdot \frac{1}{4} = \frac{1}{52} = \mathbf{P}(A \cap B)$$

Since $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$, A and B are independent events.

Question 2

Two players, **A** and **B**, are shooting at a target simultaneously and independently. For each round, the probabilities of hitting the target are:

- 1/2 for player **A**
- 1/4 for player **B**

The game continues until the target is hit (i.e. if both miss, another round is played). The game ends if, in a particular round, any of the following occur:

- **A** hits, **B** misses. **A** wins.
- **B** hits, **A** misses. **B** wins.
- **A** hits, **B** hits. **A** and **B** tie.

What is the probability of **A** winning the game?

Let A_i be the event that player **A** hits the target in the i^{th} round. Let B_i be the event that player **B** hits the target in the i^{th} round.

$$\begin{aligned}\mathbf{P}\left(\mathbf{A} \text{ wins in first round}\right) &= \mathbf{P}\left(A_1 \cap B_1^c\right) \\ &= \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}\end{aligned}$$

$$\begin{aligned}\mathbf{P}\left(\mathbf{A} \text{ wins in second round}\right) &= \mathbf{P}\left(A_1^c \cap B_1^c \cap A_2 \cap B_2^c\right) \\ &= \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{3}{4} = \frac{9}{64}\end{aligned}$$

$$\vdots$$

$$\begin{aligned}\mathbf{P}\left(\mathbf{A} \text{ wins in } i^{\text{th}} \text{ round}\right) &= \mathbf{P}\left(A_1^c \cap B_1^c \cap A_2^c \cap B_2^c \cap \ldots \cap A_{i-1}^c \cap B_{i-1}^c \cap A_i \cap B_i^c\right) \\ &= \left(\frac{1}{2} \cdot \frac{3}{4}\right)^{i-1} \cdot \frac{1}{2} \cdot \frac{3}{4}\end{aligned}$$

In other words, for a player to win in the i^{th} round, there must be $i-1$ rounds where no player won. Therefore the probability of player **A** winning is computed as:

$$\begin{aligned}\mathbf{P}\left(\text{Player } \mathbf{A} \text{ wins}\right) &= \sum_{i=1}^{\infty} \mathbf{P}\left(\text{Player } \mathbf{A} \text{ wins in } i^{\text{th}} \text{ round}\right) \\ &= \sum_{i-1=0}^{\infty} \left(\frac{1}{2} \cdot \frac{3}{4}\right)^{i-1} \cdot \frac{1}{2} \cdot \frac{3}{4} \\ &= \frac{3}{5} \quad \text{(Using geometric series formula)}\end{aligned}$$

Similarly, we have:

$$\begin{aligned}\mathbf{P}\left(\text{Player } \mathbf{B} \text{ wins}\right) &= \sum_{i=1}^{\infty} \mathbf{P}\left(\text{Player } \mathbf{B} \text{ wins in } i^{\text{th}} \text{ round}\right) \\ &= \sum_{i-1=0}^{\infty} \left(\frac{1}{2} \cdot \frac{3}{4}\right)^{i-1} \cdot \frac{1}{2} \cdot \frac{1}{4} \\ &= \frac{1}{5}\end{aligned}$$

$$\begin{aligned}
\mathbf{P}\left(\text{Game ends in a tie}\right) &= \sum_{i=1}^{\infty} \mathbf{P}\left(\text{Tie in } i^{\text{th}} \text{ round}\right) \\
&= \sum_{i-1=0}^{\infty} \left(\frac{1}{2} \cdot \frac{3}{4}\right)^{i-1} \cdot \frac{1}{2} \cdot \frac{1}{4} \\
&= \frac{1}{5}
\end{aligned}$$

Question 3

A student is taking an exam that has a one hour time limit. Suppose the probability that the student finishes the exam in less than x hours is $x/2$, for all $0 < x < 1$. Then, given that the student is still working after 0.75 hours, what is the conditional probability that the full hour is used?

Let W be the amount of time taken, in hours, by the student to finish the test.

$$\begin{aligned}
\mathbf{P}\left(W \geq 1 \,|\, W \geq 0.75\right) &= \frac{\mathbf{P}\left((W \geq 1) \cap (W \geq 0.75)\right)}{\mathbf{P}\left(W \geq 0.75\right)} \\
&= \frac{\mathbf{P}\left(W \geq 1\right)}{\mathbf{P}\left(W \geq 0.75\right)} \\
&= \frac{1 - \mathbf{P}\left(W < 1\right)}{1 - \mathbf{P}\left(W < 0.75\right)} \\
&= \frac{1 - 1/2}{1 - 3/8} \\
&= 8/10 \;=\; 0.80
\end{aligned}$$

Question 4

Suppose we have five boxes, each containing four balls:

- Box 0: 4 white balls
- Box 1: 3 white balls, 1 black ball
- Box 2: 2 white balls, 2 black balls
- Box 3: 1 white ball, 3 black balls
- Box 4: 4 black balls

(a) A box is chosen at random and two balls are drawn **without replacement**. What is the probability that both balls are black?

Since we are drawing without replacement, the only boxes where we can draw two black balls are boxes 2, 3, and 4.

$$\mathbf{P}\left(\text{Two black} \,|\, \text{Box } 2\right) \;=\; \frac{2 \cdot 1}{4 \cdot 3} \;=\; \frac{2}{12} \;=\; \frac{1}{6}$$

$$\mathbf{P}\left(\text{Two black} \,|\, \text{Box } 3\right) = \frac{3 \cdot 2}{4 \cdot 3} = \frac{6}{12} = \frac{1}{2}$$

$$\mathbf{P}\left(\text{Two black} \,|\, \text{Box } 4\right) = \frac{4 \cdot 3}{4 \cdot 3} = 1$$

Using the Law of Total Probability,

$$\begin{aligned}
\mathbf{P}\left(\text{Two black}\right) &= \sum_{k=2}^{4} \mathbf{P}\left(\text{Two black} \,|\, \text{Box } k\right) \cdot \mathbf{P}\left(\text{Box } k\right) \\
&= \left(\frac{1}{6} + \frac{1}{2} + 1\right) \cdot \frac{1}{5} \\
&= \frac{1}{3}
\end{aligned}$$

(b) A box is chosen at random and two balls are drawn **with replacement**. What is the probability that both balls are black?

Now drawing with replacement, we can draw two black balls from boxes 1, 2, 3, and 4.

$$\mathbf{P}\left(\text{Two black} \,|\, \text{Box } 1\right) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

$$\mathbf{P}\left(\text{Two black} \,|\, \text{Box } 2\right) = \left(\frac{2}{4}\right)^2 = \frac{4}{16}$$

$$\mathbf{P}\left(\text{Two black} \,|\, \text{Box } 3\right) = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

$$\mathbf{P}\left(\text{Two black} \,|\, \text{Box } 4\right) = \left(\frac{4}{4}\right)^2 = \frac{16}{16}$$

Using the Law of Total Probability,

$$\begin{aligned}
\mathbf{P}\left(\text{Two black}\right) &= \sum_{k=1}^{4} \mathbf{P}\left(\text{Two black} \,|\, \text{Box } k\right) \cdot \mathbf{P}\left(\text{Box } k\right) \\
&= \left(\frac{1}{16} + \frac{4}{16} + \frac{9}{16} + \frac{16}{16}\right) \cdot \frac{1}{5} \\
&= \frac{3}{8}
\end{aligned}$$

(c) A box is chosen at random and two balls are drawn **without replacement**. Given that both balls are black, what is the probability that they came from Box 2?

From the conditional probability formula, we have the following relationship:

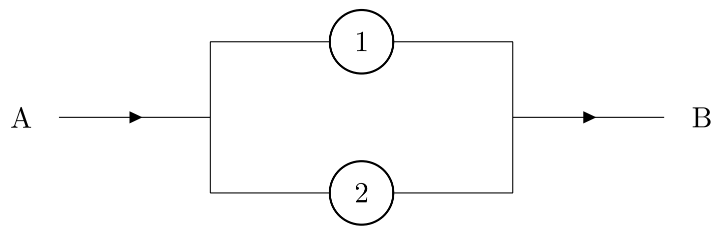
$$\mathbf{P}\left(A \,|\, B\right) \cdot \mathbf{P}\left(B\right) = \mathbf{P}\left(A \cap B\right) = \mathbf{P}\left(B \cap A\right) = \mathbf{P}\left(B \,|\, A\right) \cdot \mathbf{P}\left(A\right)$$

provided that $\mathbf{P}\left(A\right)$ and $\mathbf{P}\left(B\right)$ are non-zero.

Using the values obtained in (a),

$$\begin{aligned}
\mathbf{P}(\text{Box 2} \,|\, \text{Two black}) &= \frac{\mathbf{P}(\text{Box 2} \cap \text{Two black})}{\mathbf{P}(\text{Two black})} \\
&= \frac{\mathbf{P}(\text{Two black} \,|\, \text{Box 2}) \cdot \mathbf{P}(\text{Box 2})}{\mathbf{P}(\text{Two black})} \\
&= \frac{(1/6) \cdot (1/5)}{(1/3)} \\
&= \frac{1}{10}
\end{aligned}$$

Question 5



(a) Suppose we have a system as above that functions if at least one of its components are functioning. It is known that:

- Component 1 functions 90% of the time
- Component 2 functions 80% of the time
- The states of the components are independent

What is the probability that the system is non-functioning?

We can calculate the probability that the system is functioning using the inclusion-exclusion principle. Let C_1 and C_2 be the events that component 1 and component 2 function, respectively.

$$\begin{aligned}
\mathbf{P}(C_1 \cup C_2) &= \mathbf{P}(C_1) + \mathbf{P}(C_2) - \mathbf{P}(C_1 \cap C_2) \\
&= 0.90 + 0.8 - (0.9 * 0.8) \\
&= 0.98
\end{aligned}$$

Since we are interested in the probability that the system is non-functioning,

$$\mathbf{P}((C_1 \cup C_2)^c) = 1 - \mathbf{P}(C_1 \cup C_2) = 1 - 0.98 = 0.02$$

Alternatively, we could have used DeMorgan's Laws and the fact that if two events are independent then their complements are also independent:

$$\mathbf{P}((C_1 \cup C_2)^c) = \mathbf{P}(C_1^c \cap C_2^c) = \mathbf{P}(C_1^c) \cdot \mathbf{P}(C_2^c) = 0.1 * 0.2 = 0.02$$

(b) Now consider a system similar to (a) but with n parallel components. Once again, the system will function if at least one of its components are functioning, and the states of the components are independent.

Let C_i denote the event that component i is functioning, $i = 1, 2, \ldots, n$. What is the probability that the system is functioning?

Similar to (a), we can use the inclusion-exclusion principle. For large n, this may get tedious. It might be easier to use the second approach used in (a).

$$\begin{aligned}
\mathbf{P}\left(C_1 \cup C_2 \cup \ldots \cup C_{n-1} \cup C_n\right) &= 1 - \mathbf{P}\left(\left(C_1 \cup C_2 \cup \ldots \cup C_{n-1} \cup C_n\right)^c\right) \\
&= 1 - \mathbf{P}\left(C_1^c \cap C_2^c \cap \ldots \cap C_{n-1}^c \cap C_n^c\right) \\
&= 1 - \prod_{i=1}^{n} \mathbf{P}\left(C_i^c\right)
\end{aligned}$$