Tutorial 2

October 1, 2020

Question 1

A random variable X takes on values 1, 2, 3, 4 such that

$$2\mathbf{P}(X=1) = 3\mathbf{P}(X=2) = \mathbf{P}(X=3) = 5\mathbf{P}(X=4)$$

Find the probability distribution function and cumulative distribution function of X.

Let $\mathbf{P}(X=3) = k$.

$$2\mathbf{P}(X=1) = k \quad \Rightarrow \quad \mathbf{P}(X=1) = \frac{k}{2}$$

$$3\mathbf{P}(X=2) = k \quad \Rightarrow \quad \mathbf{P}(X=2) = \frac{k}{3}$$

$$5\mathbf{P}(X=4) = k \quad \Rightarrow \quad \mathbf{P}(X=4) = \frac{k}{5}$$

We know that:

$$\sum_{i=1}^{4} p(x_i) = 1$$

Then:

$$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1 \quad \Rightarrow \quad k = \frac{30}{61}$$

We can write the probability distribution function as:

$$f(x) = \mathbf{P}(X=x) = \begin{cases} 15/61 & x = 1 \\ 10/61 & x = 2 \\ 30/61 & x = 3 \\ 6/61 & x = 4 \\ 0 & \text{otherwise} \end{cases}$$

The cumulative distribution function can be found as:

$$F(x) = \mathbf{P}(X \leq x) = \begin{cases} 0 & x < 1 \\ 15/61 & 1 \leq x < 2 \\ 25/61 & 2 \leq x < 3 \\ 55/61 & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

Question 2

A contestant on a quiz show is presented with two questions, labelled 1 and 2, which he is to attempt in whichever order he chooses.

- If he attempts question i first, he will be allowed to proceed to question j ($j \neq i$) only if his answer to question i is correct.
- If his answer to i is incorrect, he is not allowed to answer question j.

The contestant is to receive V_i dollars for correctly answering question i, $i = 1, 2$. For example, the contestant will receive $V_1 + V_2$ dollars if he answers both questions correctly.

Let E_i, $i = 1, 2$ be the event that the contestant knows the answer to question i, and that the two events are independent. Let P_i, $i = 1, 2$ represent the respective probabilities of these events. In order for the contestant to maximize his expected winnings, which question should he attempt first?

If question 1 is attempted first, the winnings will be:

$$\begin{array}{ccc} 0 & \text{with probability} & 1 - P_1 \\ V_1 & \text{with probability} & P_1(1 - P_2) \\ V_1 + V_2 & \text{with probability} & P_1 P_2 \end{array}$$

The expected winnings if question 1 is attempted first are:

$$V_1 \cdot P_1(1 - P_2) \; + \; (V_1 + V_2) \cdot P_1 P_2$$

If question 2 is attempted first, the winnings will be:

$$\begin{array}{ccc} 0 & \text{with probability} & 1 - P_2 \\ V_2 & \text{with probability} & P_2(1 - P_1) \\ V_2 + V_1 & \text{with probability} & P_2 P_1 \end{array}$$

The expected winnings if question 2 is attempted first are:

$$V_2 \cdot P_2(1 - P_1) \; + \; (V_2 + V_1) \cdot P_2 P_1$$

The second term of both expected winnings are the same. Therefore question 1 should be attempted first if

$$V_1 \cdot P_1(1 - P_2) \; \geq \; V_2 \cdot P_2(1 - P_1)$$

or equivalently,

$$\frac{V_1 \cdot P_1}{1 - P_1} \geq \frac{V_2 \cdot P_2}{1 - P_2}$$

Example: The player has a 0.60 probability of answering question 1 correctly, worth \$200. The player has a 0.80 probability of answer question 2 correctly, worth \$100. Then:

$$\frac{V_1 \cdot P_1}{1 - P_1} \overset{?}{\geq} \frac{V_2 \cdot P_2}{1 - P_2}$$

$$\frac{200 \cdot 0.60}{0.40} \overset{?}{\geq} \frac{100 \cdot 0.80}{0.20}$$

$$300 \not\geq 400$$

The player should attempt question 2 first instead.

Question 3

(a) A particles moves n steps on a number line. The particle starts at 0, and at each step moves one unit to the right or the left, with equal probabilities. Assume all steps are independent. Let Y be the particle's position after n steps. Find the PMF of Y.

Consider each step to be a Bernoulli trial, where right is considered a success and left is considered a failure. If we let the random variable X represent the number of steps the particle takes to the right in n independent Bernoulli trials, then

$$X \sim \text{Bin}(n,\, p = 1/2)$$

If $X = j$, then the particle has taken j steps to the right and $n - j$ steps to the left, giving a final position of

$$j - (n - j) \;=\; 2j - n$$

The particle's position after n steps, Y, can be represented as a (one-to-one) function of X, using $Y = 2X - n$. Since the possible values of X were $\{0, 1, 2, \ldots, n\}$, the possible values for Y are $\{-n,\, 2-n,\, 4-n,\, \ldots,\, n\}$. The PMF of Y is

$$\mathbf{P}\left(Y = k\right) \;=\; \mathbf{P}\left(2X - n = k\right) \;=\; \mathbf{P}\left(X \;=\; (n+k)/2\right) \;=\; \binom{n}{\frac{n+k}{2}}\left(\frac{1}{2}\right)^n$$

where k is an integer in $\{-n,\, 2-n,\, 4-n,\, \ldots,\, n\}$, and $n + k$ is even. Otherwise, $\mathbf{P}\left(Y = k\right) = 0$.

(b) Using the results from (a), let D be the particle's distance from the origin after n steps. Assume that n is even. Find the PMF of D.

Let the distance from the origin after n steps, for n even, be $D = |Y|$. D is not a one-to-one function of Y. Although the event $D = 0$ is the same as $Y = 0$, for $k = 2,\, 4,\, \ldots,\, n$ we have that

$$D = k \;\equiv\; \{Y = -k\} \cup \{Y = k\}$$

The PMF of D is:

$$\mathbf{P}\left(D = 0\right) \;=\; \binom{n}{\frac{n}{2}}\left(\frac{1}{2}\right)^n$$

$$\begin{aligned}\mathbf{P}\left(D = k\right) &= \mathbf{P}\left(Y = -k\right) + \mathbf{P}\left(Y = k\right) \\ &= 2\mathbf{P}\left(Y = k\right) && \text{(By symmetry)} \\ &= 2\binom{n}{\frac{n+k}{2}}\left(\frac{1}{2}\right)^n\end{aligned}$$

Question 4

(a) If X takes values from $\{0, 1, 2, \ldots, n\}$ ($n = \infty$ is OK), show that

$$\mathbf{E}\left(X\right) \;=\; \sum_{k=1}^{n} \mathbf{P}\left(X \geq k\right)$$

Let p_k be shorthand for $\mathbf{P}\left(X = k\right)$.

$$\mathbf{E}\left(X\right) = \sum_{k=0}^{n} k \cdot p_k$$

$$= (0 \cdot p_0) + (1 \cdot p_1) + (2 \cdot p_2) + \ldots + (n \cdot p_n)$$

$$= \begin{array}{ccccc} p_1 & & & & \\ +p_2 & +p_2 & & & \\ +p_3 & +p_3 & +p_3 & & \\ \vdots & \vdots & \vdots & & \\ +p_n & +p_n & +p_n & \ldots & +p_n \end{array}$$

$$= \mathbf{P}(X \geq 1) + \mathbf{P}(X \geq 2) + \mathbf{P}(X \geq 3) + \ldots + \mathbf{P}(X \geq n)$$

$$= \sum_{k=1}^{n} \mathbf{P}(X \geq k)$$

Example with geometric distribution:

If $X \sim \text{Geom}(p)$, we know that $\mathbf{P}(X = k) = pq^{k-1}$ for $k \geq 1$ ($k-1$ failures followed by a success), and $\mathbf{E}(X) = 1/p$.

$$\begin{aligned} \mathbf{P}(X \geq k) &= \sum_{j=k}^{\infty} pq^{j-1} \\ &= p\left(q^{k-1} + q^{k+1-1} + q^{k+2-1} + \ldots\right) \\ &= p\left(q^{k-1} + q^{(k-1)+1} + q^{(k-1)+2} + \ldots\right) \\ &= pq^{k-1}\left(q^0 + q^1 + q^2 + \ldots\right) \\ &= pq^{k-1} \sum_{i=0}^{\infty} q^i \\ &= \frac{pq^{k-1}}{1-q} \\ &= q^{k-1} && \text{(Since } p = 1 - q\text{)} \end{aligned}$$

$$\mathbf{E}(X) = \sum_{k=1}^{\infty} \mathbf{P}(X \geq k) = \sum_{k=1}^{\infty} q^{k-1} = \sum_{k-1=0}^{\infty} q^{k-1} = \frac{1}{1-q} = \frac{1}{p}$$

(b) Four dice are rolled. Let M be the minimum of the four numbers. Find $\mathbf{E}(M)$.

Let X_k be the number rolled on the k^{th} die, $k = 1, 2, 3, 4$. For $c \in \{1, 2, \ldots, 6\}$, we have that

$$\mathbf{P}(X_k \geq c) = \frac{6 - c + 1}{6}$$

Then the probability of the minimum of the four numbers being greater than some c is:

$$\begin{aligned} \mathbf{P}(M \geq c) &= \mathbf{P}(X_1 \geq c \cap X_2 \geq c \cap X_3 \geq c \cap X_4 \geq c) && \text{(Definition of minimum)} \\ &= \mathbf{P}(X_1 \geq c) \cdot \mathbf{P}(X_2 \geq c) \cdot \mathbf{P}(X_3 \geq c) \cdot \mathbf{P}(X_4 \geq c) && \text{(Independence)} \end{aligned}$$

$$= \left(\frac{6 - c + 1}{6} \right)^4$$

By the results of (a),

$$\begin{aligned} \mathbf{E}(M) &= \sum_{c=1}^{6} \mathbf{P}(M \geq c) \\ &= \left(\frac{6}{6}\right)^4 + \left(\frac{5}{6}\right)^4 + \left(\frac{4}{6}\right)^4 + \left(\frac{3}{6}\right)^4 + \left(\frac{2}{6}\right)^4 + \left(\frac{1}{6}\right)^4 \\ &= \frac{2275}{1296} \approx 1.755 \end{aligned}$$