Tutorial 5

October 22, 2020

Question 1

A discrete random variable N is uniformly distributed on $\{1, 2, 3, \ldots, 10\}$.

Let X be the indicator of the event $\{N \leq 5\}$.

Let Y be the indicator of the event $\{N \text{ is even }\}$.

(a) Are X and Y independent?

(b) Find $\mathbf{E}\left((X+Y)^2\right)$.

Question 2

13 cards are drawn at random without replacement from an ordinary deck of playing cards. If X is the number of spades in these 13 cards, find the PMF of X. If, in addition, Y is the number of hearts in these 13 cards, find the probability $\mathbf{P}(X = 2, Y = 5)$. What is the joint PMF of X and Y?

Question 3

Consider the multinomial distribution:

- $m \geq 2$ categories
- $n \geq 1$ items chosen at random, **with** replacement
- $p_k = \mathbf{P}(\text{Item of type } k \text{ chosen}),\ k = 1, \ldots, m$
- $X_k = \text{Number of type } k \text{ chosen},\ k = 1, \ldots, m$

(a) Compute $\mathbf{P}(X_1 = x_1,\, X_2 = x_2,\, \ldots,\, X_m = x_m)$.

(b) Find the marginal distribution of X_k for each k. Are X_i and X_j independent?

Question 4

Let $(X_1, X_2, X_3) \sim \text{Multi}(n, p_1, p_2, p_3)$. Find the conditional distribution of X_1 given that $X_3 = x_3$. Intuitively, we expect that

$$X_1 \,|\, X_3 = x_3 \sim \text{Binomial}\left(n - x_3,\, \frac{p_1}{p_1 + p_2}\right)$$

Question 5

Suppose $X \sim \text{Bin}(N, p)$, where the number of trials, N, is also a random variable (but independent of the trials themselves). Then conditioned on the fact that $N = n$, the number of successes, X, would have distribution $\text{Bin}(n, p)$. What can be said about the unconditional distribution of X, in particular the case when N is a Poisson random variable?

Question 6

Following the setup of the previous question, let $Y = N - X$ represent the number of failures. It is implied that Y has distribution $\text{Poisson}(\lambda \cdot (1 - p))$. Show that X and Y are independent. [Note that this is strongly due to the Poisson distribution of N, and does not happen otherwise (i.e. with deterministic N).]