# Tutorial 6

November 5, 2020

We will do Questions 2 and 4 next time!

## Question 1

Determine the value of $C$ such that:

$$f(x,y) = \begin{cases} C\,(x+y) & 0 < x < 3, \quad x < y < x+2 \\ 0 & \text{otherwise} \end{cases}$$

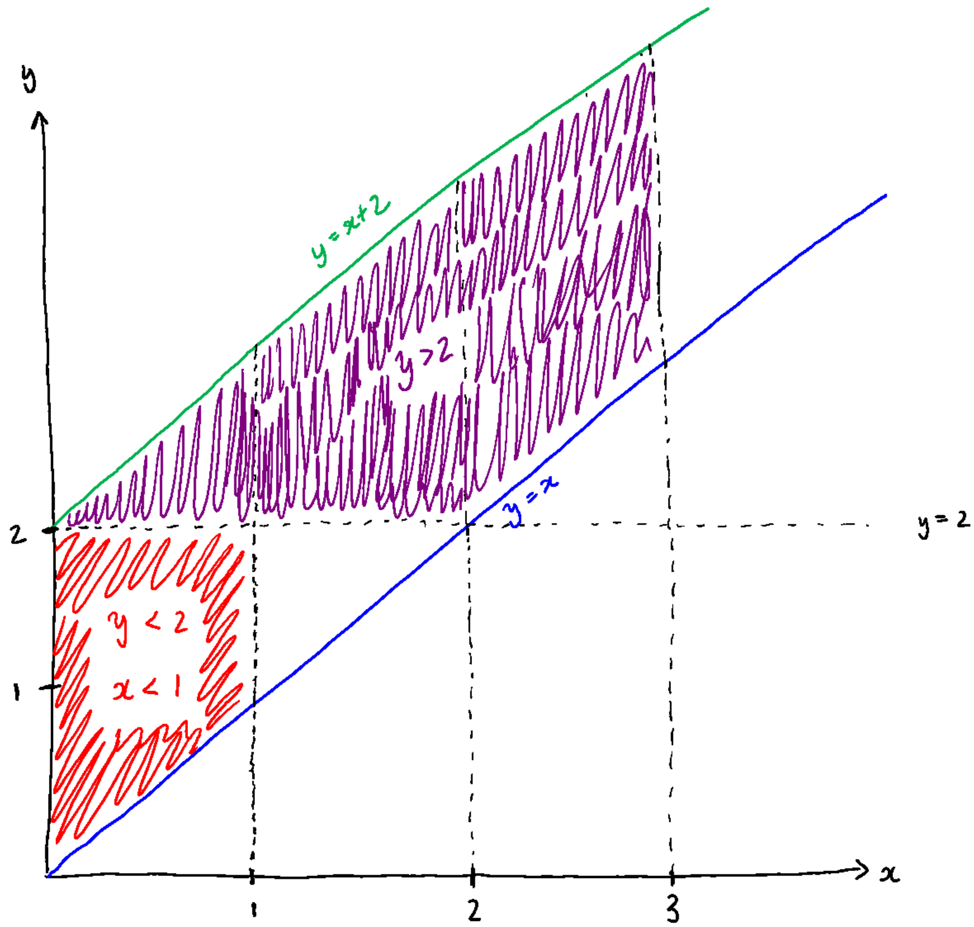
is a valid joint PDF.

We know that for $f(x,y)$ to be a valid joint PDF, if we integrate it over its support we should get 1.

$$\begin{aligned}
\int_0^3 \int_x^{x+2} C(x+y)\,dy\,dx &= C \int_0^3 \int_x^{x+2} x+y\,dy\,dx \\
&= C \int_0^3 xy + \frac{1}{2}y^2 \Big|_{y=x}^{y=x+2} dx \\
&= C \int_0^3 4x+2\,dx \\
&= C\left(2x^2+2x\right)\Big|_{x=0}^{x=3} \\
&= 24C
\end{aligned}$$

Since $24C = 1$, it follows that $C = 1/24$.

To compute the following probabilities, we will reference this diagram:




(a) $\mathbf{P}(X < 1, Y < 2)$

$$\frac{1}{24}\int_0^1\int_x^2 x + y\,dy\,dx = \frac{1}{24}\int_0^1 xy + \frac{1}{2}y^2\Big|_{y=x}^{y=2}\,dx$$

$$= \frac{1}{24}\int_0^1 -\frac{3}{2}x^2 + 2x + 2\,dx$$

$$= \frac{1}{24}\left(-\frac{1}{2}x^3 + x^2 + 2x\right)\Big|_{x=0}^{x=1}$$

$$= \frac{1}{24}\left(-\frac{1}{2} + 1 + 2\right)$$

$$= \frac{5}{48}$$

(b) $\mathbf{P}(Y > 2)$

We will break up the purple region into two parts. The first integral is for the triangular region, the second integral is for the parallelogram.

$$\frac{1}{24}\int_0^2\int_2^{x+2} x + y\,dy\,dx + \frac{1}{24}\int_2^3\int_x^{x+2} x + y\,dy\,dx$$

$$= \frac{1}{24}\int_0^2 xy + \frac{1}{2}y^2\Big|_{y=2}^{y=x+2} dx + \frac{1}{24}\int_2^3 xy + \frac{1}{2}y^2\Big|_{y=x}^{y=x+2} dx$$

$$= \frac{1}{24}\int_0^2 \frac{3}{2}x^2 + 2x\,dx + \frac{1}{24}\int_2^3 4x + 2\,dx$$

$$= \frac{1}{24}\left(\frac{1}{2}x^3 + x^2\right)\Big|_{x=0}^{x=2} + \frac{1}{24}\left(2x^2 + 2x\right)\Big|_{x=2}^{x=3}$$

$$= \frac{5}{6}$$

(c) $\mathbf{E}(X)$

$$\mathbf{E}(X) = \frac{1}{24}\int_0^3\int_x^{x+2} x(x+y)\,dy\,dx$$

$$= \frac{1}{24}\int_0^3 x\int_x^{x+2} x + y\,dy\,dx$$

$$= \frac{1}{24}\int_0^3 x(4x+2)\,dx$$

$$= \frac{1}{24}\int_0^3 4x^2 + 2x\,dx$$

$$= \frac{1}{24}\left(\frac{4}{3}x^3 + x^2\right)\Big|_{x=0}^{x=3}$$

$$= \frac{36+9}{24} = \frac{45}{24}$$

It should be noted that for $g(X)$, a function solely of $X$,

$$\mathbf{E}\left(g(X)\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)f(x,y)\,dy\,dx = \int_{-\infty}^{\infty} g(x)\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_{-\infty}^{\infty} g(x)f_X(x)\,dx$$

A similar result holds for $h(Y)$, a function solely of $Y$.

# Question 3

The joint density of $X$ and $Y$ is given by:

$$f(x,y) = \begin{cases} e^{-(x+y)} & 0 < x < \infty, \quad 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the density of the random variable $X/Y$.

We start by finding the CDF of $X/Y$.

$$\begin{aligned}
F_{X/Y}(c) &= \mathbf{P}\left(X/Y \le c\right) \\
&= \iint\limits_{x/y \le c} e^{-(x+y)}\,dx\,dy \\
&= \int_0^{\infty}\int_0^{cy} e^{-x}\cdot e^{-y}\,dx\,dy \\
&= \int_0^{\infty} e^{-y}\left(1 - e^{-cy}\right)\,dy \\
&= \left(-e^{-y} + \frac{e^{-(c+1)y}}{c+1}\right)\Bigg|_{y=0}^{y=\infty} \\
&= 1 - \frac{1}{c+1}
\end{aligned}$$

$$F_{X/Y}(c) = \begin{cases} 0 & c < 0 \\ 1 - \dfrac{1}{c+1} & c > 0 \end{cases}$$

Differentiating to find the density, we obtain:

$$f_{X/Y}(c) = \begin{cases} \dfrac{1}{(c+1)^2} & 0 < c < \infty \\ 0 & \text{otherwise} \end{cases}$$