Tutorial 7

November 12, 2020

Question 1

Let X and Y be independent $N(0,1)$ distributed random variables. Show that $X+Y$ and $X-Y$ are independent $N(0,2)$ distributed random variables.

Let $U = X+Y$ and $V = X-Y$. Solving for X and Y, we obtain:

$$X = \frac{U+V}{2} \quad Y = \frac{U-V}{2}$$

The Jacobian of this transformation is:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\det(\mathbf{J}) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}, \quad |\det(\mathbf{J})| = \frac{1}{2}$$

The joint density of U and V can be found as:

$$\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \cdot |\det(\mathbf{J})| \\
&= f_X\left(\frac{u+v}{2}\right) \cdot f_Y\left(\frac{u-v}{2}\right) \cdot |\det(\mathbf{J})| && \text{(Since } X \perp Y\text{)} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{u+v}{2}\right)^2\right\} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{u-v}{2}\right)^2\right\} \cdot \frac{1}{2} \\
&= \frac{1}{\sqrt{2\pi \cdot 2}} \frac{1}{\sqrt{2\pi \cdot 2}} \exp\left\{\left(-\frac{1}{2}\right)\left(\frac{u^2+2uv+v^2}{2^2} + \frac{u^2-2uv+v^2}{2^2}\right)\right\} \\
&= \frac{1}{\sqrt{2\pi \cdot 2}} \frac{1}{\sqrt{2\pi \cdot 2}} \exp\left\{-\frac{1}{2}\left(\frac{u^2}{2}\right) - \frac{1}{2}\left(\frac{v^2}{2}\right)\right\} \\
&= \frac{1}{\sqrt{2\pi \cdot 2}} \exp\left\{-\frac{1}{2}\left(\frac{u^2}{2}\right)\right\} \cdot \frac{1}{\sqrt{2\pi \cdot 2}} \exp\left\{-\frac{1}{2}\left(\frac{v^2}{2}\right)\right\}
\end{aligned}$$

for $-\infty < u < \infty$ and $-\infty < v < \infty$.

Based on the form of the PDF we can conclude that:

- $U = X+Y \sim N(0,2)$

- $V = X - Y \sim N(0, 2)$
- $U \perp V$

By the properties of the normal distribution, we already knew that U and V would have the distributions above. The key point of this exercise was to show that U and V would also be independent.

Question 2

The joint PDF of X and Y is given by:

$$f(x,y) = \begin{cases} e^{-(x+y)} & x > 0, \quad y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the PDF of $U = \dfrac{X+Y}{2}$.

$$\text{Let } U = \frac{X+Y}{2} \quad \text{and} \quad V = Y$$

Then $X = 2U - V$ and $Y = V$.

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\det(\mathbf{J}) = (2)(1) - (0)(-1) = 2, \quad |\det(\mathbf{J})| = 2$$

The joint PDF of U, V is:

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(2u - v, v) \cdot |\det(\mathbf{J})| \\ &= e^{-(2u-v+v)} \cdot 2 \\ &= 2e^{-2u} \end{aligned}$$

What is the support?

$$x > 0 \quad \Rightarrow \quad 2u - v > 0 \quad \Rightarrow \quad 2u > v$$

$$y > 0 \quad \Rightarrow \quad v > 0$$

$$2u > 0 \quad \Rightarrow \quad u > 0 \quad \text{and} \quad 0 < v < 2u$$

$$f_{U,V}(u,v) = \begin{cases} 2e^{-2u} & u > 0, \quad 0 < v < 2u \\ 0 & \text{otherwise} \end{cases}$$

To find the marginal distribution of U, we should integrate with respect to V.

$$f_U(u) = \int_0^{2u} 2e^{-2u}\, dv = 4ue^{-2u}$$

for $u > 0$, and zero otherwise.

Question 3

Suppose that two random variables X_1 and X_2 have the following joint distribution:

$$f(x_1, x_2) = \begin{cases} 4x_1x_2 & 0 < x_1 < 1, \quad 0 < x_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the joint pdf of the new random variables

$$Y_1 = \frac{X_1}{X_2} \quad Y_2 = X_1X_2$$

What is the marginal density of Y_1?

$$Y_1Y_2 = \frac{X_1}{X_2}X_1X_2 = X_1^2 \quad \Rightarrow \quad X_1 = (Y_1Y_2)^{\frac{1}{2}} = Y_1^{\frac{1}{2}}Y_2^{\frac{1}{2}}$$

$$\frac{Y_2}{Y_1} = \frac{X_1X_2}{\frac{X_1}{X_2}} = X_2^2 \quad \Rightarrow \quad X_2 = \left(\frac{Y_2}{Y_1}\right)^{\frac{1}{2}} = Y_1^{-\frac{1}{2}}Y_2^{\frac{1}{2}}$$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{2}y_1^{-\frac{1}{2}}y_2^{\frac{1}{2}} & \dfrac{1}{2}y_1^{\frac{1}{2}}y_2^{-\frac{1}{2}} \\ -\dfrac{1}{2}y_1^{-\frac{3}{2}}y_2^{\frac{1}{2}} & \dfrac{1}{2}y_1^{-\frac{1}{2}}y_2^{-\frac{1}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{2}\left(\dfrac{y_2}{y_1}\right)^{\frac{1}{2}} & \dfrac{1}{2}\left(\dfrac{y_1}{y_2}\right)^{\frac{1}{2}} \\ -\dfrac{1}{2}\left(\dfrac{y_2}{y_1^3}\right)^{\frac{1}{2}} & \dfrac{1}{2}\left(\dfrac{1}{y_1y_2}\right)^{\frac{1}{2}} \end{bmatrix}$$

$$\begin{aligned}
\det(\mathbf{J}) &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{y_2}{y_1}\right)^{\frac{1}{2}}\left(\frac{1}{y_1y_2}\right)^{\frac{1}{2}} - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{y_2}{y_1^3}\right)^{\frac{1}{2}}\left(\frac{y_1}{y_2}\right)^{\frac{1}{2}} \\
&= \frac{1}{4}\cdot\frac{1}{y_1} + \frac{1}{4}\cdot\frac{1}{y_1} \\
&= \frac{1}{2y_1}
\end{aligned}$$

What is the new support?

$$x_1 > 0 \quad \Rightarrow \quad (y_1 y_2)^{\frac{1}{2}} > 0$$

This means that either $y_1,\, y_2 > 0$, or $y_1,\, y_2 < 0$. But $y_1,\, y_2 \not< 0$ since x_1 and x_2 were both positive. So it must be that

$$y_1 > 0 \quad \text{and} \quad y_2 > 0 \tag{$*$}$$

$$x_1 < 1 \quad \Rightarrow \quad (y_1 y_2)^{\frac{1}{2}} < 1 \quad \Rightarrow \quad y_2 < \frac{1}{y_1}$$

$$x_2 > 0 \quad \Rightarrow \quad \left(\frac{y_2}{y_1}\right)^{\frac{1}{2}} > 0$$

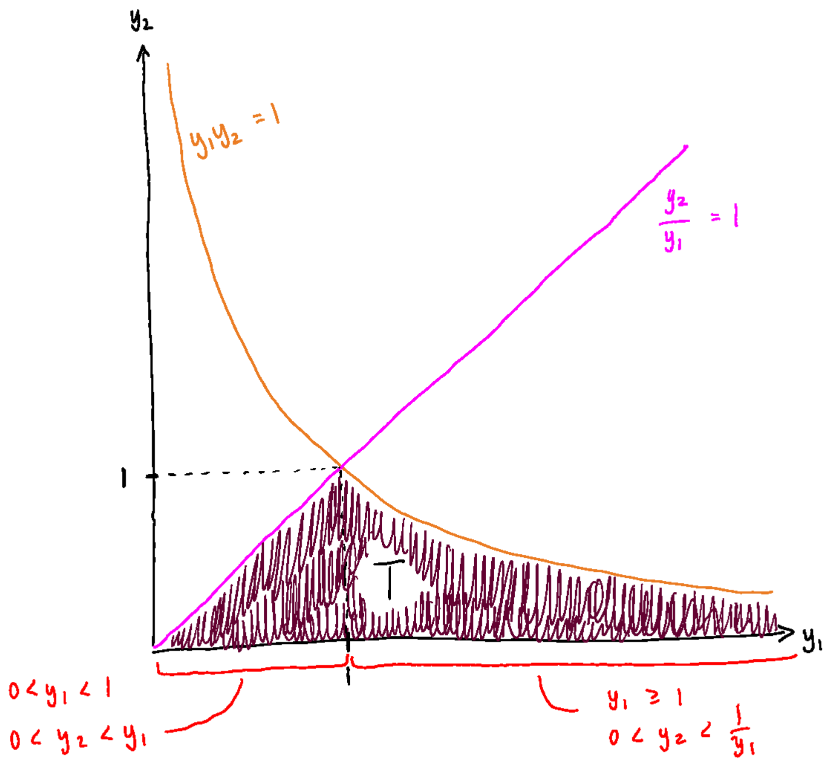
By the same reasoning as $(*)$, it must be that

$$y_1 > 0 \quad \text{and} \quad y_2 > 0$$

$$x_2 < 1 \quad \Rightarrow \quad \left(\frac{y_2}{y_1}\right)^{\frac{1}{2}} < 1 \quad \Rightarrow \quad y_2 < y_1$$

Therefore, the new support is:

$$\mathcal{T} = \left\{(y_1,\, y_2) \,:\, y_1 > 0, \quad 0 < y_2 < \min\left\{y_1,\, \frac{1}{y_1}\right\}\right\}$$



The new joint PDF is:

$$g(y_1, y_2) = f\left((y_1 y_2)^{\frac{1}{2}}, \left(\frac{y_2}{y_1}\right)^{\frac{1}{2}}\right) \cdot |\det(\mathbf{J})|$$

$$= 4 \cdot y_2 \cdot \left|\frac{1}{2y_1}\right|$$

$$= 2\left(\frac{y_2}{y_1}\right) \qquad \text{(Since } y_1 > 0\text{)}$$

for $(y_1, y_2) \in \mathcal{T}$, and zero otherwise.

To find the marginal density of Y_1, we should integrate out Y_2.

$$g(y_1) = \begin{cases} \int_0^{y_1} 2\left(\frac{y_2}{y_1}\right) dy_2 & 0 < y_1 < 1 \\ \int_0^{1/y_1} 2\left(\frac{y_2}{y_1}\right) dy_2 & y_1 \geq 1 \end{cases}$$

$$= \begin{cases} y_1 & 0 < y_1 < 1 \\ \frac{1}{y_1^3} & y_1 \geq 1 \end{cases}$$

(and zero otherwise.)

Question 4

Continuing from Question 3, find the marginal of

$$Z_1 = \frac{X_1}{X_2}$$

by first transforming to Z_1 as above, and $Z_2 = X_1$, and then integrating z_2 out of the joint pdf.

$$X_1 = Z_2$$

$$X_2 = \frac{X_1}{Z_1} = \frac{Z_2}{Z_1} = Z_1^{-1} Z_2$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -z_1^{-2} z_2 & z^{-1} \end{bmatrix}$$

$$\det(\mathbf{J}) = \frac{z_2}{z_1^2}$$

What is the new support?

$$x_1,\, x_2 > 0 \quad \Rightarrow \quad z_1 \;=\; \frac{x_1}{x_2} > 0$$

$$x_1 > 0 \quad \Rightarrow \quad z_2 > 0$$
$$x_1 < 1 \quad \Rightarrow \quad z_2 < 1$$

Combining the two, we get $0 < z_2 < 1$.

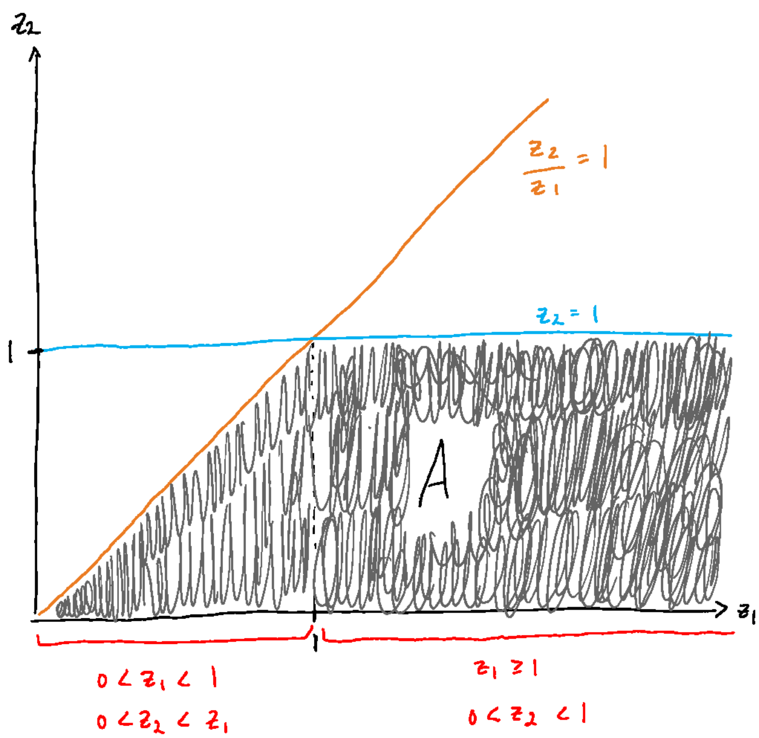
$$x_2 > 0 \quad \Rightarrow \quad \frac{z_2}{z_1} > 0$$
$$x_2 < 1 \quad \Rightarrow \quad \frac{z_2}{z_1} < 1$$

Combining the two, we get

$$0 < \frac{z_2}{z_1} < 1 \quad \Rightarrow \quad 0 < z_2 < z_1 \qquad (z_1 > 0)$$

This means that $0 < z_2 < \min\{1,\, z_1\}$. The new support is:

$$\mathcal{A} \;=\; \{(z_1,\, z_2) \,:\, z_1 > 0, \quad 0 < z_2 < \min\{1,\, z_1\}\}$$



The new joint PDF is:

$$
\begin{aligned}
h(z_1,\, z_2) &= f\left(z_2,\, \frac{z_2}{z_1}\right) \cdot |\det(\mathbf{J})| \\
&= 4\, z_2\, \frac{z_2}{z_1} \cdot \left|\frac{z_2}{z_1^2}\right| \\
&= 4\left(\frac{z_2^3}{z_1^3}\right)
\end{aligned}
$$

for $(z_1,\, z_2) \in \mathcal{A}$, and zero otherwise.

To find the marginal density of Z_1, we need to integrate out Z_2.

$$
\begin{aligned}
h(z_1) &= \begin{cases} \int_0^{z_1} 4\left(\frac{z_2^3}{z_1^3}\right)\, dz_2 & 0 < z_1 < 1 \\ \int_0^{1} 4\left(\frac{z_2^3}{z_1^3}\right)\, dz_2 & z_1 \geq 1 \end{cases} \\
&= \begin{cases} z_1 & 0 < z_1 < 1 \\ \frac{1}{z_1^3} & z_1 \geq 1 \end{cases}
\end{aligned}
$$

(and zero otherwise.)