Tutorial 8

November 19, 2020

Question 1

Consider an infinite series of Bernoulli(p) trials.

Let H_1 be the number of successes in trials 1 - 50.

Let H_2 be the number of successes in trials 51 - 100.

Let H_3 be the number of successes in trials 101 - 150.

(a) Find the covariance between X, the number of successes in trials 1 - 100, and Y, the number of successes in trials 51 - 150.

$$X = H_1 + H_2 \qquad Y = H_2 + H_3$$

$$\begin{aligned}
\mathbf{Cov}(X, Y) &= \mathbf{Cov}(H_1 + H_2, H_2 + H_3) \\
&= \mathbf{Cov}(H_1, H_2) + \mathbf{Cov}(H_1, H_3) + \mathbf{Cov}(H_2, H_2) + \mathbf{Cov}(H_2, H_3) \\
&= 0 + 0 + \mathbf{Var}(H_2) + 0 \\
&= np(1-p) = 50p(1-p)
\end{aligned}$$

because H_2 has distribution Binomial($n = 50$, p).

(b) Find the covariance between X, the number of successes in trials 1 - 100, and Z, the number of failures in trials 51 - 150.

$$Z := 100 - Y = 100 - H_2 - H_3$$

$$\begin{aligned}
\mathbf{Cov}(X, Z) &= \mathbf{Cov}(H_1 + H_2, 100 - H_2 - H_3) \\
&= -\mathbf{Cov}(H_1, H_2) - \mathbf{Cov}(H_1, H_3) - \mathbf{Cov}(H_2, H_2) - \mathbf{Cov}(H_2, H_3) \\
&= -0 - 0 - \mathbf{Var}(H_2) - 0 \\
&= -50p(1-p)
\end{aligned}$$

(c) Find the correlation between Y and Z.

$$\begin{aligned}
\mathbf{Cov}(Y, Z) &= \mathbf{Cov}(Y, 100 - Y) \\
&= -\mathbf{Cov}(Y, Y) \\
&= -\mathbf{Var}(Y)
\end{aligned}$$

$$= -100p(1-p)$$

because $Y = H_2 + H_3$ has distribution Binomial($n = 100$, p).

$$\mathbf{Var}(Z) = \mathbf{Var}(100 - Y) = \mathbf{Var}(Y)$$

$$\mathbf{Corr}(Y, Z) = \frac{\mathbf{Cov}(Y, Z)}{\sqrt{\mathbf{Var}(Y)\,\mathbf{Var}(Z)}} = \frac{-\mathbf{Var}(Y)}{\sqrt{\mathbf{Var}(Y)\,\mathbf{Var}(Y)}} = -1$$

Question 2

X and Y are independent random variables with probability density functions:

$$f_X(x) = \begin{cases} 4ax & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 4by & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the correlation coefficient between $X + Y$ and $X - Y$.

We start by finding the values of a and b. Since X and Y are independent, integrating each marginal density over its respective support should equal 1.

$$\int_0^1 4ax\,dx = 2ax^2\Big|_{x=0}^{x=1} = 2a = 1 \quad \Rightarrow \quad a = \frac{1}{2}$$

Similarly, we will obtain $b = \dfrac{1}{2}$.

We also require the variances of X and Y.

$$\mathbf{E}(X) = \int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_{x=0}^{x=1} = \frac{2}{3}$$

$$\mathbf{E}(X^2) = \int_0^1 2x^3\,dx = \frac{1}{2}x^4\Big|_{x=0}^{x=1} = \frac{1}{2}$$

Similarly, $\mathbf{E}(Y) = \dfrac{2}{3}$ and $\mathbf{E}(Y^2) = \dfrac{1}{2}$.

$$\mathbf{Var}(X) = \mathbf{E}(X^2) - (\mathbf{E}(X))^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{9}{18} - \frac{8}{18} = \frac{1}{18}$$

Similarly, $\mathbf{Var}(Y) = \dfrac{1}{18}$.

Let $U = X + Y$ and $V = X - Y$.

$$\mathbf{Cov}(U, V) = \mathbf{Cov}(X + Y, X - Y)$$

$$= \mathbf{Cov}(X, X) - \mathbf{Cov}(X, Y) + \mathbf{Cov}(Y, X) - \mathbf{Cov}(Y, Y)$$
$$= \mathbf{Var}(X) - 0 + 0 - \mathbf{Var}(Y)$$
$$= 0$$

Since the covariance of U and V is zero, the correlation is automatically zero.

Question 3

Let $U \sim \text{Unif}(-1, 1)$ and $V = 2|U| - 1$.

(a) Find the distribution of V.

A $\text{Unif}(-1, 1)$ random variable will have CDF:

$$F_U(u) = \frac{u - (-1)}{1 - (-1)} = \frac{u + 1}{2}$$

for $-1 \le u \le 1$, $F_U(u) = 0$ for $u < -1$, and $F_U(u) = 1$ for $u > 1$.

$$\begin{aligned}
\mathbf{P}(|U| \le u) &= \mathbf{P}(-u \le U \le u) \\
&= \mathbf{P}(U \le u) - \mathbf{P}(U \le -u) \\
&= \frac{u+1}{2} - \left(\frac{-u+1}{2}\right) \\
&= \frac{u + 1 + u - 1}{2} \\
&= \frac{2u}{2} \\
&= u \quad (0 \le u \le 1)
\end{aligned}$$

Therefore $|U| \sim \text{Unif}(0, 1)$. Then

$$2|U| \sim \text{Unif}(0, 2)$$

and

$$V = 2|U| - 1 \sim \text{Unif}(-1, 1)$$

(b) Show that U and V are uncorrelated but not independent. This example illustrates that knowing the marginal distributions of two random variables does not determine the joint distribution.

The density of $U \sim \text{Unif}(-1, 1)$ is:

$$f_U(u) = \frac{1}{1 - (-1)} = \frac{1}{2}$$

for $-1 \le u \le 1$, and zero otherwise.

The expected value of a $\text{Unif}(a, b)$ random variable is $\frac{1}{2}(a + b)$.

$$\mathbf{Cov}(U, V) = \mathbf{E}(UV) - \mathbf{E}(U)\,\mathbf{E}(V)$$

$$= \mathbf{E}\left(U\left(2|U|-1\right)\right) - 0 \cdot 0$$

$$= \mathbf{E}\left(2U \cdot |U| - U\right)$$

$$= 2\mathbf{E}\left(U \cdot |U|\right) - \mathbf{E}\left(U\right)$$

$$= 2\mathbf{E}\left(U \cdot |U|\right)$$

$$= 2\left(\int_{-1}^{1} |u| \cdot u \cdot f_U(u)\, du\right)$$

$$= 2\left(-\int_{-1}^{0} u^2 \cdot \frac{1}{2}\, du + \int_{0}^{1} u^2 \cdot \frac{1}{2}\, du\right)$$

$$= 0$$

Since the covariance of U and V is zero, the correlation will also be zero. However,

$$\mathbf{P}\left(V \geq 0 \,|\, U = 0\right) = 0 \neq \mathbf{P}\left(V \geq 0\right) = \frac{1}{2}$$

which shows that U and V are not independent despite their covariance being zero.

Question 4

Pick a point uniformly distributed in the triangle $x \geq 0$, $y \geq 0$, $x + y \leq 1$. Compute

The joint density of X and Y is:

$$f_{X,Y}(x,y) = \begin{cases} c & x,y \geq 0, \quad x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Before we can start this question, we need to find the value of c and the individual marginal densities.

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{0}^{1}\int_{0}^{1-x} c\, dy\, dx$$

$$= c\int_{0}^{1} (1-x)\, dx$$

$$= c\left(-\frac{(1-x)^2}{2}\right)\Bigg|_{x=0}^{x=1}$$

$$= \frac{c}{2}$$

c must equal 2.

$$f_X(x) \;=\; \int_0^{1-x} 2\,dy \;=\; 2(1-x), \quad 0 \le x \le 1$$

$$f_Y(y) \;=\; \int_0^{1-y} 2\,dx \;=\; 2(1-y), \quad 0 \le y \le 1$$

The conditional densities are:

$$f_{X\,|\,Y=y}(x) \;=\; \frac{f_{X,Y}(x,y)}{f_Y(y)} \;=\; \frac{2}{2(1-y)} \;=\; \frac{1}{1-y}, \quad 0 < x < 1-y$$

$$f_{Y\,|\,X=x}(y) \;=\; \frac{f_{X,Y}(x,y)}{f_X(x)} \;=\; \frac{2}{2(1-x)} \;=\; \frac{1}{1-x}, \quad 0 < y < 1-x$$

(a) $\mathbf{E}\,(X\,|\,Y=y)$

$$\begin{aligned}
\mathbf{E}\,(X\,|\,Y=y) &= \int_{-\infty}^{\infty} x \cdot f_{X\,|\,Y=y}(x)\,dx \\
&= \int_0^{1-y} x \cdot \frac{1}{1-y}\,dx \\
&= \left(\frac{x^2}{2(1-y)}\right)\Bigg|_{x=0}^{x=1-y} \\
&= \frac{(1-y)^2}{2(1-y)} \\
&= \frac{1-y}{2}
\end{aligned}$$

(b) $\mathbf{E}\,(Y\,|\,X=x)$

By a similar calculation,

$$\mathbf{E}\,(Y\,|\,X=x) \;=\; \frac{1-x}{2}$$

We can also verify that the law of total expectation holds.

$$\mathbf{E}\,(X) \;=\; \int_0^1 2x(1-x)\,dx \;=\; x^2 - \frac{2}{3}x^3\Bigg|_{x=0}^{x=1} \;=\; \frac{1}{3} \;=\; \mathbf{E}\,(Y)$$

$$\mathbf{E}\,(X) \;=\; \mathbf{E}\,(\mathbf{E}\,(X\,|\,Y)) \;=\; \mathbf{E}\left(\frac{1-Y}{2}\right) \;=\; \frac{1}{2} - \frac{1}{2}\mathbf{E}\,(Y)$$

$$\frac{1}{3} \;=\; \frac{1}{2} - \frac{1}{2}\left(\frac{1}{3}\right) \;=\; \frac{3}{6} - \frac{1}{6} \;=\; \frac{2}{6} \;=\; \frac{1}{3}$$